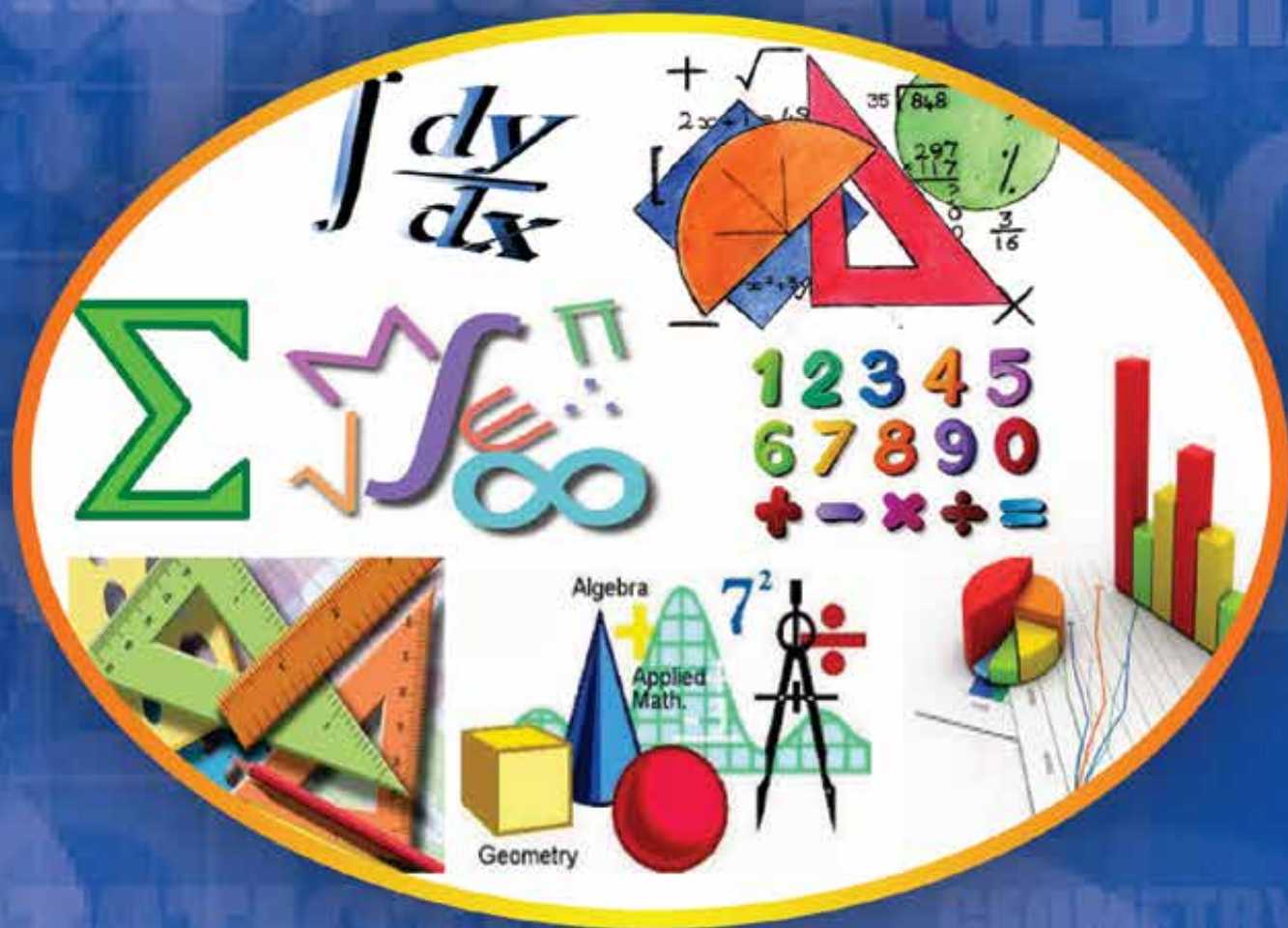


TRIGONOMETRY

YEAR 11

MATHEMATICS

FUNCTIONS PROBABILITY
CALCULUS ALGEBRA



PRODUCED BY
CURRICULUM DEVELOPMENT UNIT
MINISTRY OF EDUCATION
SUVA, FIJI

GEOMETRY
STATISTICS
GRAPHS



MATHEMATICS FOR YEAR 11

CURRICULUM DEVELOPMENT UNIT
MINISTRY OF EDUCATION
FIJI
2015





© Ministry of Education, Fiji 2015

The Curriculum Advisory Services (CAS) section of the Ministry of Education owns the copyright to this text book, YEAR 11 MATHEMATICS. Schools may reproduce this in part or full for classroom purposes only.

Acknowledgement of the CAS section of the Ministry of Education copyright must be included on any reproductions.

Any other use of this book must be referred to the Permanent Secretary of Education through Director CAS, Ministry of Education.

Published By

Ministry of Education

Marella House

Private Mail Bag

Suva

Fiji

Tel (679) 3313050



ACKNOWLEDGEMENT

This Year 11 Mathematics text book is the pioneering attempt by Curriculum Development Unit for a mathematics text for upper secondary Mathematics.

This book has been compiled and developed by **Shiek Mohammed Masud, Senior Education Officer Mathematics at Curriculum Development Unit.**

This is the first time a Mathematics textbook has been developed for upper secondary by Curriculum Development Unit.

The following teachers are acknowledged for their contribution towards the development of this textbook:

Mr Amitesh Chandra	Exams and Assessment Unit
Mrs Nazmin Nisha	Nasinu Muslim College
Mr Shaheem Ali	Nayala High School.
Ms Abhu Keshni Lal	Lelean Memorial School
Mr Nilesh Chand	Nasinu Secondary School
Mrs Shaleshni Prasad	Lelean Memorial School

Pierce, Rod. "Math is Fun - Maths Resources" Math Is Fun. Ed. Rod Pierce. 3 Nov 2014. 11 Jan 2015 <<http://www.mathsisfun.com/>>

<http://regentsprep.org>



PREFACE

This textbook is the first mathematics textbook developed by CDU for Year 11. It is based on the new syllabus of 2015 which has been revised after two decades.

This book tries to capture the depth of only Year 11 Mathematics content based on the new syllabus.

This textbook is aimed at complimenting the implementation of the revised syllabus and act as a resource for teachers in preparation of their lessons.

Year 11 Mathematics textbook is expected to act as a bridging resource from year10 and establish a strong foundation for higher secondary mathematics in Year 12 and Year 13.

Students are encouraged to keep referring to the notes and illustrations in the book for better understanding.

Teachers and students are advised to use other resources as well for enhancing of teaching and learning.

TABLE OF CONTENT

STRAND

1



BASIC MATHEMATICS 1

Basic Number Theory 1

Order of Operation 10

Formula Manipulation 15

Measurement 19

STRAND

2



ALGEBRA 31

Manipulation of Algebraic Expressions 31

Expanding and Factorising 36

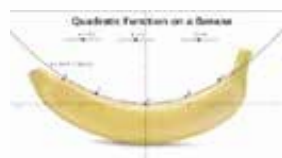
Equations and In - Equation 44

Sequences 55

Matrices 64

STRAND

3



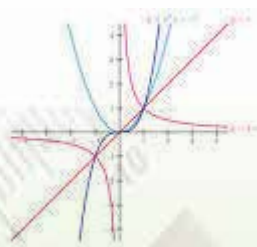
FUNCTIONS 74

Domain and Range 76

Inverse 79

STRAND

4



GRAPHS 83

Linear Graph 83

Simultaneous Equations 84

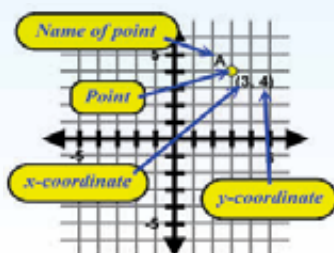
Quadratic Graph 93

Cubic Graph 98

Graph of a Circle 104

Exponential Graph 107



STRAND**5****COORDINATE GEOMETRY 111**

Writing coordinates	111
Distance Formula	113
Mid-point Formula	114
Equation of a line	115
Parallel Lines	118
Perpendicular Lines	119

STRAND**6****TRIGONOMETRY 121**

Trigonometric Ratios	121
Pythagoras Theorem	126
Trigonometric Graphs	127
Angles in Radians	130
Angles in Quadrants	138
Solving Trigonometric Equations	142
Exact Values	144

STRAND**7****STATISTICS 146**

Data	147
Measures of Central Tendency	154
Measures of Spread	159
Histogram & Bar Graph	160
Cumulative Frequency	162

STRAND**8****PROBABILITY 166**

Equally Likely	167
Sample Space	168
Probability Tree	169

STRAND**9****CALCULUS 175**

Differentiation	176
Integration	187

$$\frac{d}{dx} \int_a^x f(t) = f(x)$$

STRAND 1

BASIC MATHEMATICS

LEARNING OUTCOMES

Sub Strand

Basic Number Theory

- Study and use properties of binary operations
- Explain and work out closure, identity and inverse element of a system
- Use calculator and to work with properties of zero.

- Manipulate Binary operations
- Identify Binary properties
- Determine Closure of a set
- Find identity and Inverse elements
- Know properties of zero
- Use order of operation
- Identify everyday measurements like length, area, volume, mass, capacity, money and time

BINARY OPERATION

A **binary operation** on A is a rule that assigns to every pair of elements of A a unique element of A .

We are used to addition and multiplication of real numbers and these are examples of binary operations.

Formally, a binary operation is a function. For example, addition on the integers could be defined as the function $+$ ($A + B \rightarrow C$) with our familiar way of producing the answer. Thus $+(2,3) = 5$ and $+(-4,11) = 7$. This looks a little too weird and we are probably happier to see $2+3=5$ and $-4 + 11 = 5$. Likewise, if $*$ is a binary operation on a set then we will write $a*b$ rather than $*(a,b)$.

The idea of a binary operation is just a way to produce an element of a set from a given pair of elements of the same set. In the case of a finite set we could list the rule in a table.

In studying binary operations on sets, we tend to be interested in those operations that have certain properties



Identity Element

Let A be a set on which there is a binary operation. An element e of this set is called an

identity, e , if for all $a \in A$

$$e \cdot a = a$$

For example **0** is an identity for the usual **addition** on the real numbers.

Given a binary operation on a set there might be no identity element. We tend to be familiar with the situation in which there is a unique identity. Also note that an identity for one operation does not have to be an identity for another operation. Think of addition and multiplication on the reals where the identities are 0 and 1 respectively.

Being an identity is a global property in the sense that it must work for **ALL** elements of the set.

Associative Property

A binary operation $*$ is said to be *associative* if for all elements a , b and c satisfy :

$$(a*b)*c = a*(b*c)$$

For convenience let's drop the symbol for the operation and just write **$(ab)c = a(bc)$** . The associative property then allows us to speak of abc without having to worry about whether we should find the answer to ab first and then that answer "multiplied" by c rather than evaluate bc first and then "multiply" a with that answer. Whichever way we process the expression we end up with the same element of the set. Note though that it does not say we can do the product in any order (i.e. ab and ba may not have the same value).

Commutative Properties

A binary operation $*$ is *commutative* if

$$a * b = b * a$$

for **ALL** possible a and b in the set. Addition and multiplication in the reals are commutative operations whereas multiplication of matrices generally is not.

Note that the definition requires $ab = ba$ for all pairs of elements. That some element commutes with all elements does not make the operation commutative.

For a finite set whose binary operation is given in a table, it is easy to observe whether the operation is commutative. If we write the elements along the rows in the same order as down the column then if the table is **symmetric about the diagonal** (i.e reflexive in the leading diagonal) then it is **commutative**.

Distributive Properties

Suppose there are two binary operations $*$ and $\#$ defined on a set A . We say that $*$

distributes over $\#$ if for all a, b and c in A ,

$$a * (b \# c) = a * b \# a * c \quad \text{and} \quad (a \# b) * c = a * b \# a * c.$$

Addition over real numbers is associative.

Other examples of distributive operations are set union and intersection.

Inverse

If identities exist for a certain binary operation then we can talk about inverses (should they exist). If e is an identity then a is an *inverse* of b if

$$a * b = b * a = e$$

Also we need to point out that an element might have more than one inverse or for that matter, no inverse. Certainly if identities do not exist then it does not make sense to talk about inverses.

Even if there is an identity then elements may not have an inverse.

So binary operation simply means any operation between **two** objects.

It is simply a rule for combining two objects of a given type, to obtain another object of that type. You first learned of binary operations in primary school. The objects you were using were numbers and the binary operations you investigated were addition, subtraction, multiplication and division. As you will discover in this text, binary operations need not be applied only to numbers.

CLOSURE OF A SET

A set is *closed* (under an operation) if and only if the operation on two elements of the set produces another element of the **same** set. If an element outside the set is produced, then the operation is *not closed*.

Example: If you multiply two *real numbers*, you will get another real number. Since this process is always true, it is said that the real numbers are "closed under the operation multiplication". There is simply no way to escape the set of real numbers when multiplying.

Closure: When you combine any two elements of the set, the result is also included in the set.

Example: If you add two even numbers (from the set of even numbers),
is the sum even?

Checking:

$$10 + 14 = 24$$

Yes, 24 is even.

$$6 + 10 = 16$$

Yes, 16 is even.

$$8 + 200 = 208$$

Yes, 208 is even.

Since the sum (the answer) is always even, the set of even numbers is **closed**
under the operation of addition.

Does this mean
the even numbers
are closed for all
operations?

Let's check out this question. If you divide two even
numbers (from the set of even numbers), is the quotient
(the answer) even?

Checking: $20 \div 2 = 10$ Yes, 10 is even.

$24 \div 2 = 12$ Yes, 12 is even.

$30 \div 2 = 15$ **NO, 15 is not even!**

When you find even ONE example that does not work, the set is not closed under that operation. The even numbers are **not closed** under division.

The elements in a binary table are displayed horizontally and

vertically outside the table (in this table, the elements are 1, 2, 3, and 4).

If the elements inside the table are limited to the elements 1, 2, 3, and 4,

the table is **closed** under the indicated operation.

	1	2	3	4
1	4	3	2	1
2	3	1	4	2
3	2	4	1	3
4	1	2	3	4

Reading the table:

Read the first value from the left hand column and the second value from the top row. The answer is in the cell where the row and column intersect.

For example, $a * b = b$, $b * b = c$, $c * d = b$, $d * b = a$.

Example 2

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

What is the identity element for the operation $*$?

(What single element will always return the original value?)

The identity element is **a**.

$a * a = a$, $b * a = b$, $c * a = c$,

$d * a = d$

This table shows the operation $*$. The operation is working on the finite set $A = \{a, b, c, d\}$. The table shows the 16 possible calculations using the elements of set A .

Checking for the Identity Element:

You will know the identity element when you see it, because all of the values in its row or column are the same as the row or column headings.

What is the **inverse element** for b ?

(What element, when paired with b , will return the identity element a ?)

The inverse element of b is d . $b * d = a \rightarrow b^{-1} = d$

Is the operation $*$ commutative?

(Does the property $ab = ba$ hold for

ALL possible arrangements of values?)

Start testing values:

$a * b = b * a$ is true since both sides equal b .

$c * d = d * c$ is true since both sides equal b .

WOW! Having to test ALL possible arrangements could take forever! There must be an easier way.....

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Testing for Commutativity Shortcut:

It is easy to check whether an operation defined by a table is commutative. Simply draw a diagonal line from upper left to lower right, and see if the table is symmetric about this line.

Since this table is symmetric about the diagonal (from upper left to lower right), the operation is commutative. It would only have taken one instance of lack of commutativity for this answer to have been "no".

The operation $*$ is *commutative*.

True or False:

$$(a * b) * c = a * (b * c) ???$$

Perform the operations in the order indicated by the parentheses:

$$(a * b) * c = a * (b * c)$$

This question deals with only one case of the **associative property** for this table. Unfortunately **there is NO shortcut for checking associativity** as there is for checking commutativity when working with a table.

If asked the general question "Does this operation possess the associative property?", you would have to check ALL possible arrangements. On the other hand, if you find one instance where associativity fails, you are done and the answer is "NO".

EXERCISE

1.1

- Which property is illustrated by the equation $ma + mb = m(a + b)$
A . Associative B. Commutative C. Distributive D. Identity
- If A and X represent integers, $A + X = X + A$ is an example of which property?
A . Associative B. Commutative C. Distributive D. Identity
- Operation $\#$ is defined on the set $\{m, a, t, h\}$ as shown in the table below:
 - Is this operation commutative?
 - Name the identity element, or explain why none exists.
 - For each element having an inverse, name the element and its inverse.
 - True or false:

$$(m \# a) \# t = m \# (a \# t)$$

#	m	a	t	h
m	h	m	a	t
a	m	a	t	h
t	a	t	h	m
h	t	h	m	a

4. Operation $\square$ is defined on the set $\{a, b, c, d\}$ shown in the table below:

- Is this operation commutative?
- Name the identity element, or explain why none exists.
- For each element having an inverse, name the element and its inverse.
- True or false:

$\square$	a	b	c	d
a	d	c	a	b
b	c	a	b	d
c	a	b	d	c
d	b	d	c	a

$$d \square (c \square b) = (d \square c) \square b$$

5. Operation $\square$ is defined on the set $\{1, 2, 3, 4\}$ as shown in the table below

- Is this operation commutative?
- Name the identity element, or explain why none exists.
- For each element having an inverse, name the element and its inverse.

$\square$	1	2	3	4
1	4	3	2	1
2	3	1	4	2
3	2	4	1	3
4	1	2	3	4

- True or false: $(1 \square 2) \square 3 = 1 \square (2 \square 3)$

6. Operation $\#$ is defined on the set $\{\text{blue star}, \text{light blue star}, \text{red sphere}, \text{dark blue sphere}\}$ as shown in the table below:

- Is this operation commutative?
- Name the identity element, or explain why none exists.
- For each element having an inverse, name the element and its inverse.
- True or false:

$\#$	blue star	light blue star	red sphere	dark blue sphere
blue star	red sphere	dark blue sphere	blue star	light blue star
light blue star	dark blue sphere	blue star	light blue star	red sphere
red sphere	blue star	light blue star	red sphere	dark blue sphere
dark blue sphere	light blue star	red sphere	dark blue sphere	blue star

$$\text{blue star} \# (\text{light blue star} \# \text{dark blue sphere}) = (\text{blue star} \# \text{light blue star}) \# \text{dark blue sphere}$$

7. Given: $a \blacklozenge b = 2a - 3b$

What is the value of $-2 \blacklozenge 5$?

8. Given $x \blackstar y = \frac{1}{2}b - 3a$, find the value of $-2 \blackstar 6$.

9. An operation $*$ is defined on a set of real numbers as $a * b = 3a^2 - 5b$.

Find the value of

[i] $2 * 3$

[ii] $4 * -2$

[iii] $-5 * -3$

Test Your Knowledge

Which of the following equations follow the concept of commutative property?

1.  +  =  +  (A)
 +  +  (B)
 +  (C)
 +  (D)
 None of the above (E)
2.  +  =  +  (A)
 +  (B)
 +  (C)
 +  (D)
 None of the above (E)

ORDER OF OPERATION

When there is more than one operation involved in a mathematical problem, it must be solved by using the correct order of operations. Remember, calculators will perform operations in the order which you enter them; therefore, **you will need to enter the operations in the correct order for the calculator to give you the right answer.**

* In Mathematics, the order in which mathematical problems are solved is extremely important.

Rules

1. Calculations must be done from left to right.
2. Calculations in brackets (parenthesis) are done first. When you have more than one set of brackets, do the inner brackets first.
3. Exponents (or radicals) must be done next.
4. Multiply and divide in the order the operations occur.
5. Add and subtract in the order the operations occur.

Remember to:

- Simplify inside groupings of parentheses, brackets and braces first. Work with the innermost pair, moving outward.
- Simplify the exponents.
- Do the multiplication and division in order from left to right.
- Do the addition and subtraction in order from left to right.

Acronyms to Help you remember

How will you remember this order?

Try the following Acronyms:

- * **Please Excuse My Dear Aunt Sera**
(**P**arenthesis, **E**xponents, **M**ultiply, **D**ivide, **A**dd, **S**ubtract)
- * **BEDMAS**
(**B**rackets, **E**xponents, **D**ivide, **M**ultiply, **A**dd, **S**ubtract)

Examples:

$12 \div 4 + 3^2$ $12 \div 4 + 9$ $3 + 9$ 12	Rule 3 : Exponent first Rule 4 : Multiply or Divide as they appear Rule 5 : Add or Subtract as they appear
$(4^2 + 5) - 3$ $21 - 3$ 18	Rule 2 : Everything in the brackets first Rule 5 : Add or Subtract as they appear
$20 \div (12 - 2) \times 3^2 - 2$ $20 \div 10 \times 3^2 - 2$ $20 \div 10 \times 9 - 2$ $18 - 2$ 16	Rule 2 : Everything in the brackets first Rule 3 : Exponents Rule 4 : Multiply and Divide as they appear Rule 5 : Add or Subtract as they appear

Does It Make a Difference? What If I Don't Use the Order of Operations?

Mathematicians were very careful when they developed the order of operations. Without the correct order, watch what happens:

$15 + 5 \times 10$ -- Without following the correct order, I know that $15+5=20$ multiplied by 10 gives me the answer of 200.

$15 + 5 \times 10$ -- Following the order of operations, I know that $5 \times 10 = 50$ plus 15 = 65. ***This is the correct answer, the above is not!***

You can see that it is absolutely critical to follow the order of operations. Some of the most frequent errors students make occur when they do not follow the order of operations when solving mathematical problems. Students can often be fluent in computational work yet do not follow procedure. Use the handy acronyms to ensure that you never make this mistake again!!

PROPERTIES OF ZERO

Addition property:

The addition property says that a number does not change when adding or subtracting zero from that number

Examples: $2 + 0 = 2$ $x + 0 = x$ $(a + b) + 0 = a + b$

Additive inverse property

If you add two numbers and the sum is zero, we call the two numbers additive inverses of each other

For example: 2 is the additive inverse of -2 because $2 + -2 = 0$
 -2 is also the additive inverse of 2 because $-2 + 2 = 0$

Multiplication property

The multiplication property says that zero times any number is equal to zero

Examples: $2 \times 0 = 0$ $265893748945798882 \times 0 = 0$
 $x \times 0 = 0$ $(x + y + z + r) \times 0 = 0$

Power of 0

When a number is raised to the power 0, we are not actually multiplying the particular number by 0. For example, let us take 2^0 . In this case we are not actually multiplying the number 2 by 0.

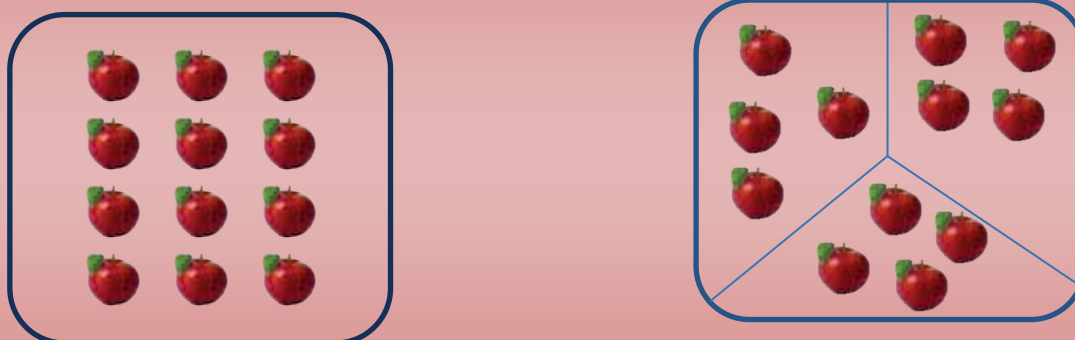
We define $2^0 = 1$, so that each power of 2 is one factor of 2 larger than the last, e.g., 1, 2, 4, 8, 16, 32... This involves the rules of exponents particularly division.

Thus, this result says that number raised to the power zero is equal to 1.

Dividing by 0

To see why we can't divide by 0, let us look at what is meant by "division": Division is splitting into equal parts or groups. It is the result of "fair sharing".

Example: there are 12 apples, and 3 friends want to share them, how do they divide the apples?



So they get 4 each: $12/3 = 4$

Now, let us try dividing the 12 apples among **zero** people, how much does each person get?

Does that question even make sense? No, of course it doesn't.

You can't share among zero people, and you can't divide by 0.

Another way of seeing it:

After dividing, can you multiply to get back again?

12 divided by 3 is 4 $\longrightarrow$ 4 times 3 is 12

12 divided by 0 is ? $\longrightarrow$? times 0 is 12

But multiplying by 0 gives 0, so that won't work.

So what is $0 \div 0$?

$0 \div 0$ is like asking "how many 0s in 0?"

Are there no zeros in zero at all? Or perhaps there is exactly one zero in zero?

Or many zeros?

So $0 \div 0$ is indeterminate (it could be any value).

When you try to divide by zero, things stop making sense therefore dividing by 0 is **undefined** !

I Wonder

Test Your Knowledge

 I know a way by which I can get a total of 120 by using five zeros 0,0,0,0,0 and any mathematical operator. Can you do this?

FORMULA MANIPULATION

A formula links one quantity to one or more other quantities. For example, $A = \pi r^2$

This formula can be used to determine A for any given value of r.

Changing the subject of a formula

Changing the subject of a formula is exactly the same as solving an equation. The key thing to remember is that '*whatever you do to on one side of the formula you must do to the other side*'.

To rearrange a formula you may

- add or subtract the same quantity to or from both sides
- multiply or divide both sides by the same quantity

We may know the area of a circle and need to find the radius. To do this, we rearrange the formula to make the radius the subject.

The area of a circle (A) is πr^2 . So: $A = \pi r^2$

We will now rearrange the formula to make 'r' the subject.

$$A = \pi r^2$$

Start by dividing both sides by π .

$$\frac{A}{\pi} = r^2$$

Then take the square root of both sides.

$$\sqrt{\frac{A}{\pi}} = r$$

Examples

1. If $F = ma$, what's the value of F when $m = 5$ and $a = 3$?

$$\begin{aligned} F &= ma \\ &= 5 \times 3 \\ &= 15 \end{aligned}$$

2. The formula for the volume (V) of a sphere is: $V = \frac{4}{3}\pi r^3$

Rearrange the formula to make 'r' the subject $V = \frac{4}{3}\pi r^3$

Multiply both sides by 3. $3V = 4\pi r^3$

Divide by 4π . $\frac{3V}{4\pi} = r^3$

Take the cube root of both sides.

$$\sqrt[3]{\frac{3V}{4\pi}} = r$$

3. **Make x the subject of the formula in $y = mx + c$**

Subtract c from both sides $y - c = mx + c - c$

Simplify $y - c = mx$

Divide both sides by $\frac{y - c}{m} = \frac{mx}{m}$

Simplify $x = \frac{y - c}{m}$

Formulae with brackets and fractions

If there are brackets included in the formula then it is easier if you expand them first. Look out for implied brackets in fractions, (part (c) below), and include them where necessary.

Make the subject of the formula in each of the following cases.

$$a(x + b) = c$$

Expand the brackets

$$ax + ab = c$$

Subtract ab from each side

$$ax = c - ab$$

Divide both sides by a

$$x = \frac{c - ab}{a}$$

EXERCISE

1.2

1 Calculate:

(a) $6 + 7 \times 2$

(b) $8 - 3 \times 2$

(c) $19 - 4 \times 3$

(d) $3 \times 6 - 9$

(e) $15 - 4 + 7 \times 2$

2. Decide whether each of the statements below is *true* or *false*.

(a) $6 \times 7 - 2 = 40$

(b) $8 \times (6 - 2) + 3 = 56$

(c) $35 - 7 \times 2 = 56$

(d) $3 + 7 \times 3 = 30$

(e) $18 - (4 + 7) = 21$

3. What is the value of this?

$$\frac{2^4 + (16 - 3 \times 4)}{(6 + 3^2) \div (7 - 4)}$$

4. What is the value of $(7 - \sqrt{9}) \times (42 - 3 + 1)$?

5. What is the value of $(33 - 9 / 3) + (4 \times 3 - 32)$?

6. What is the value of $(5^2 - 5) / (4^2 + 8 - 7 \times -5)$?

7. Make x the subject in the formulas

[i] $y = axb$

[ii] $y = w(x + t)$

[ii] $y = a(x + b) c$

[iii] $ax - y + z + b$

[iv] $ax - y = 2y$

8. In each case, make the letter at the end the subject of the formula

[i] $2s = 2ut + at^2$, (a)

[ii] $v^2 = u^2 + 2as$, (s)

[iii] $y = a^2x + b^2$, (a)

9. Make x the subject in the formulas

[i] $\frac{x}{a} = \frac{y+z}{b}$

[ii] $\frac{a(x+y)}{b} = c$

10. In each case, make the letter at the end the subject of the formula

[i] $s = ut + \frac{1}{2}at^2$, (u)

[ii] $\frac{v-u}{a} = t$, (v)

[iii] $\frac{y-x^2}{a} = 3z$, (y)

11. In each case, make the letter at the end the subject of the formula

[i] $y = \sqrt{x+3}$, (x)

[ii] $x^2 - y^2 = a^2$, (x)

[iii] $\sqrt{x^2 + y^2} = y$, (x)

[iv] $\frac{1}{m} = \frac{1}{s} + \frac{1}{t}$, (s)

12. Simplify the following

[i] $\frac{3a}{5} \cdot \frac{7ab}{3}$

[ii] $\frac{3a-1}{2b^2} \cdot \frac{3a^2b}{9a-3}$

Sub Strand

Measurement

- List everyday local measurements.
- Use measurements in calculations
- Conversion in measurements
- Determine how interest rate and length of loan affect total cost of credit
- Compare 3 different savings plan
- Show advantages of early loan or credit repayment

In the metric system, each of the common kinds of measure -- length, weight, and capacity -- has *one* basic unit of measure. To measure smaller amounts, divide the basic unit into parts of ten, a hundred, or a thousand, and so on. To measure larger amounts, multiply the basic unit by ten, a hundred, or a thousand, and so on.

Brief History

The Metric system of measurement was created about two hundred years ago by a group of French scientists to simplify measurement.

Length: 1 kilometre (km) = 1000 meters (m) 1 centimetre (cm) = .01 meter (m) 1 millimetre (mm) = .001 meter (m)	Capacity: 1 millilitre = .001 liter (l)
Weight: 1 kilogram (kg) = 1000 grams (g) 1 milligram (mg) = .001 gram (g)	<i>Kilo</i> means thousand (1000) <i>Hecto</i> means hundred (100) <i>Deca</i> means ten (10) <i>Deci</i> means one-tenth (1/10) <i>Centi</i> means one-hundredth (1/100) <i>Milli</i> means one-thousandth (1/1000)
TIME 1 hr = 60 min 1 min = 60 sec	

Conversions

Sometimes you need to convert from one unit of measure to another similar unit.

Proportions will help you make conversions when working with measurements.

Create a unit **conversion ratio**, which is always equal to 1.

Example 1: Convert 4.214 kilometres to meters.

Using a **conversion ratio**:

$$\frac{? \text{ m}}{4.214 \text{ km}} = \frac{1000 \text{ m}}{1 \text{ km}}$$

conversion ratio

Solving algebraically, the answer is **4214 m**

Another way to look at the problem:

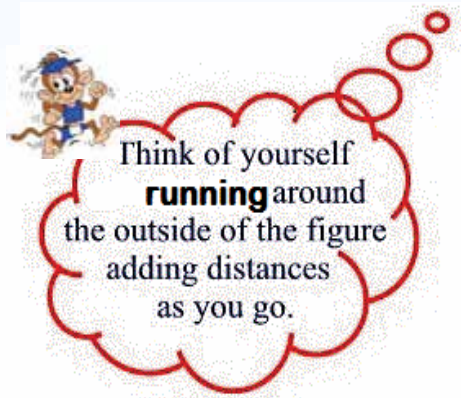
When making metric conversions, arrange the prefixes from largest to smallest:

Kilo hecto deca **UNIT** deci centi milli
(meter, gram or litre)

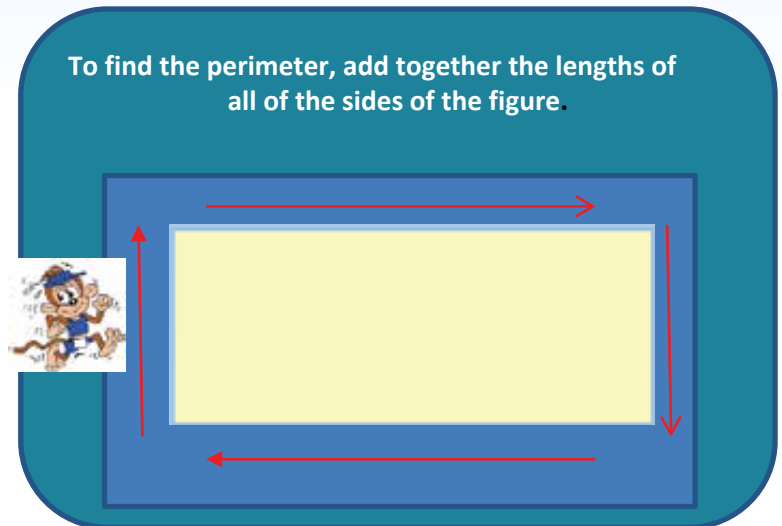
To convert 4.214 kilometres to meters, notice that in the chart above, the name **metre** is three places to the right of the name **kilo**. This implies that the decimal point in our number needs to be moved three places to the right. The answer is **4214 metres**.

PERIMETER AND CIRCUMFERENCE

Perimeter is the word used to describe the distance around the outside of a figure.



Triangle	3 sides
Quadrilateral	4 sides
Pentagon	5 sides
Hexagon	6 sides
Heptagon or Septagon	7 sides
Octagon	8 sides
Nonagon	9 sides
Decagon	10 sides
Dodecagon	12 sides



Refresh your polygon memories:

When working with perimeter, references may be made to the names of polygons. Listed at the left are some of the more common polygons whose names you should know.

Remember that "regular polygons" are polygons whose sides are all the same length and whose angles are all the same size. Not all polygons are "regular".

Circumference is the word used to describe the distance around the outside of a circle.

Like perimeter, the circumference is the distance around the outside of the figure. Unlike perimeter, in a circle there are no straight segments to measure, so a special formula is needed.



$$C = 2\pi r = \pi d$$

Use when you know the radius $C = 2\pi r$

Use when you know the diameter $C = \pi d$

Example 1

Ansaar and Muskaan are jogging around a circular track in the park.

The diameter of the track is 400m.

Find the number of kilometres they jogged if they made two complete trips around the track.

$$\begin{aligned} C &= d \times \pi \\ &= 400 \text{ m} \times \pi \\ &= 1256.6371 \end{aligned}$$



$$\begin{aligned} \text{Therefore for 2 rounds} &: 1256.6371 \times 2 \\ &= 2513.2741 \end{aligned}$$

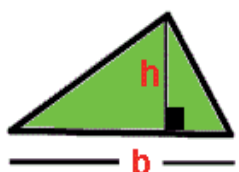
$$\text{Hence } 2513.2741 / 1000 = 2.51 \text{ km}$$

Work should be held in the calculator until the final rounding of the answer occurs.

AREA

The sum of areas of all the faces of a 3D solid figure (both plane and curved) is called its total surface area.

Area (**triangle**)



$$A = \frac{1}{2}bh$$

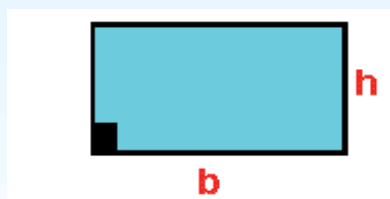
Area (**equilateral triangle**)



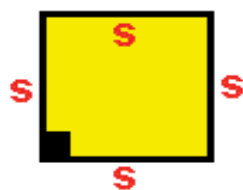
$$A = \frac{s^2 \sqrt{3}}{4}$$

or

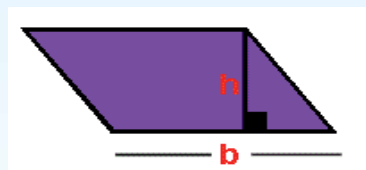
$$A = \frac{1}{2}bh$$

Area (**rectangle**)

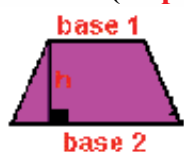
$$A = bh$$

Area (**square**)

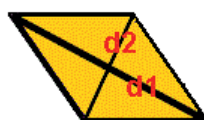
$$A = s^2 = bh$$

Area (**parallelogram**)

$$A = bh$$

Area (**trapezium**)

$$A = \frac{1}{2}h(b_1 + b_2)$$

Area (**rhombus**)

$$A = \frac{1}{2}d_1 \cdot d_2$$

$$A = bh$$

Area (**circle**)

$$A = \pi r^2$$

Area of **sectors** of circle

(Sectors are similar to "pizza slices" of a circle)

Semi-circle

(1/2 of circle = 1/2 of area)



$$A = \frac{1}{2}\pi r^2$$

Quarter-Circle

(1/4 of circle = 1/4 of area)



$$A = \frac{1}{4}\pi r^2$$

Any Sector

(fractional part of the area)

$$A = \frac{n}{360} \pi r^2$$

where n is the number of degrees in the central angle of the sector.

$$A = \frac{C_s}{2\pi r} \pi r^2$$

where C_s is the arc length of the sector.

Area (**regular polygon**)

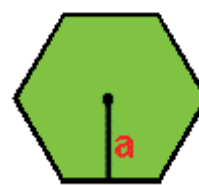
$$A = \frac{1}{2}ap$$



a = apoth



p = perimeter

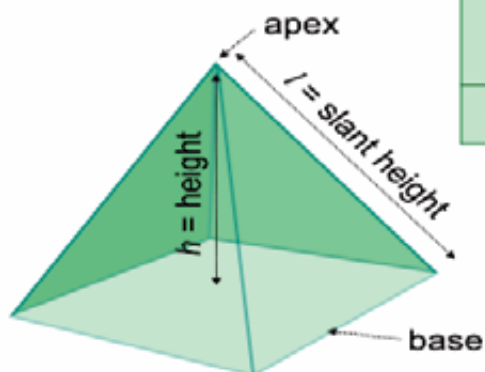


Regular polygons have all sides of equal length.

VOLUME

The total amount of space occupied by a 3D solid figure is called its volume.

Pyramid - A three-dimensional figure whose base is a polygon of any number of sides and other faces are triangles with a common vertex, called apex.



Type of Pyramid	Vertices	Edges	Surface (Faces)	
			Total	Type
	Corners of Base + 1	Sides of Base $\times 2$	Sides of Base + 1	Type of Base, Triangular
Square	5	8	5	Square, Triangular

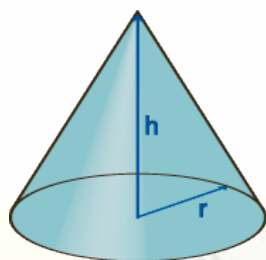
Surface Area	Area of Base + $\frac{1}{2} \times \text{Perimeter of Base} \times l$
Volume	$\frac{1}{3} \times \text{Area of Base} \times h$

Sphere - A sphere is a perfectly round three-dimensional geometrical object. It is the set of points which are all at the same distance from a given point in space.



Vertices	Edges	Surface (Faces)	
		Total	Type
0	0	1	Curved

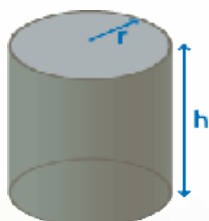
Surface Area	$4\pi r^2$
Volume	$\frac{4}{3}\pi r^3$



Cone

$$\text{Surface Area} = \pi r^2 + \pi r s$$

$$\text{Volume} = \frac{1}{3}(\pi r^2 h)$$



cylinder

$$\text{Surface Area} = 2\pi r^2 + 2\pi r h$$

$$\text{Volume} = \pi r^2 h$$

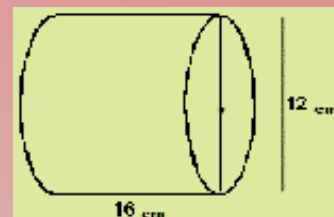
Test Your Knowledge | Three Way Mix Up

Jack has three blue tiles, three yellow tiles and three red tiles. He put them together in a square so that no two tiles of the same colour were beside each other.

Can you find another way to do it?
Can you find ALL the ways to do it?

Example 1: Consider this problem where radius is needed but not given:

- Find the volume of this cylinder.
- Find the surface area if this cylinder represents a can which has no lids.



NB:

- When a formula needs a radius, **be sure that you are working with the radius** and not the diameter. In this problem 12 cm is the diameter (radius = 6 cm).

$$V = \pi r^2 h = \pi \cdot 6^2 \cdot 16 = 1809.56 \text{ cm}^3$$

- You may need to **amend your formula** to fit a particular situation. The surface area formula for a cylinder is

$$SA = 2\pi rh + 2\pi r^2$$

This formula includes the areas of the top and bottom (which are 2 circles). If the top and bottom are **NOT** to be considered, the formula will be

$$SA = 2\pi rh = 2\pi \cdot 6 \cdot 16 = 603.19 \text{ cm}^2$$

Example 2: Consider this problem that gives "hints" on what is needed to solve the problem.

A die is a cube moulded from hard plastic. The edge of a die measures 0.62 cm.

Dice are usually produced in a mould which holds 100 die at one time.

To the *nearest* cm^3 , how much plastic is needed to fill this large mould?



When working with word problems, be sure to **read carefully to determine what the question wants you to find**. This question clearly involves volume since it states "to the nearest **cubic** cm." Also, the answer must be for 100 dice, not 1 die.

$$\text{Volume of one die} = lwh$$

$$= (.62)(.62)(.62) = 0.238 \text{ cm}^3$$

$$\text{For 100 dice} = (0.238)(100) = 23.8 = 24 \text{ cm}^3$$

Example 3: Consider this problem with different units of measure.

A concrete truck arrives at a job site holding 15m^3 of concrete. If the enclosure being constructed is 3m across and 2 thick, how long, to the *nearest m* will the enclosure be if constructed from the amount of concrete on the truck?

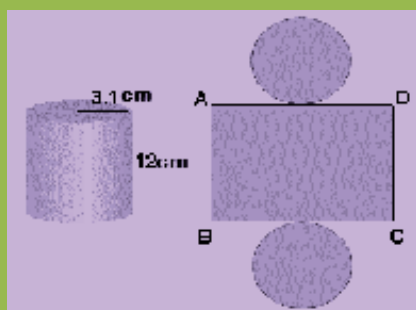
Always read carefully to determine if all of the measurements within a problem are expressed in the **same units**. This problem deals with inches, feet and cubic yards.

$$\begin{aligned} V &= lwh \\ 15 &= (l)(3)(2) \\ \text{length} &= 2.5 \text{ m} \end{aligned}$$



Example 4: Consider this problem that requires visualization. When dealing with surface area it is often helpful to **imagine the figure cut apart** (called a **net**).

In this example, imagine cutting off the top and bottom of the cylinder and then slicing the remaining shape and flattening it out.



- Find AB
- Find AD
- Find the surface area of the cylinder.

a. **$AB = 12\text{cm}$**

- b. AD = the length around the "edge" of the cylinder (which is a circle) = circumference of a circle

$$= 2\pi r = 2\pi(3.1)$$

$$= 19.48 \text{ cm}$$

- c. Surface area

$$SA = 2\pi rh + 2\pi r^2 = 294.12 \text{ cm}^2$$

Money Management

- Compare the total cost of borrowing for any purchase of an item for two different types of credit.
- Calculate the interest rate for the loan period and its annual equivalent
- Show, by examples of calculations, how the interest rate and loan length affect the cost of credit.
- Give examples to measure the total cost of credit, including the hidden costs, not the interest rate only
- Calculate and evaluate different savings and debt options and match debt with the future value of purchases.
- Demonstrate the benefits of early repayment.

Financial Education

Financial Education teaches the management of personal finances and investment given a person's personal circumstances. Financial Education will result in a future generation of financially competent young men and women leaving schools, who are able to make informed decisions and sufficiently manage their personal finances and investments vis-à-vis their own personal circumstances, whilst contributing positively to their communities, the economy and the country.

Acquiring Financial Competence involves more than the development of mathematics skills that are traditionally included in the school curriculum, such as recognition of coins and notes, and calculations involving sums of cash. Personal Financial Education promotes social inclusion and helps break the cycle of financial exclusion. Being financially excluded means being cut off from the services and benefits of the financial services industry.

Those who are financially excluded become adept at budgeting by saving bits of money in jars or envelopes. But not only does money stored like this not accrue interest, it is also vulnerable to theft.

Financially excluded households are not able to give children the experience of managing money that others take for granted. There is evidence to suggest that such children go on to become financially excluded themselves. Certainly, "children living in poorer families learn about and experience the economic world differently from their peers in other families."5 This cycle of exclusion needs to be broken if future generations are to become financially competent, it is imperative that Financial Education begins early. This means starting Financial Education as soon as students begin their primary education, building on learning throughout the primary and secondary years of schooling. Not all students complete secondary education, so Financial Education learning at class levels 1-8 and forms 3-7 in the Fiji educational system will be fundamental to realising these goals.

Fiji Financial Education Curriculum Development Project
Senior Secondary Levels, Forms Five and Six
FinED Fiji
A Manual for Teachers
Year 11 & 12

Credit is borrowing money or buying goods and paying for them later. It is the power to obtain goods and services with an obligation on the borrower of future payment to the lender (the course of credit).

Why should borrowing money incur interest to the borrower and earn interest for the lender?

Interest is a charge that is paid by the borrower for the use of someone else's money. A lender expects to be rewarded for not having access to his money.

Table below provides different types of credit, matching the lender, the likely purpose of the credit, potential fees and interest rates to the type of credit.

Type	Lender	Purpose	Fees	Interest rates
Bank loan	Retail Bank	Small items such as a car, household items	Set up fee	Depend on bank policy 12 – 20%?
Third tier lender loan	Pay day lenders	Emergency items Funeral, temporary shortfall in income to pay expenses	Combination of Set up fees Establishment fees Insurance?	Usually 22 – 26% but can be very high if a very short term loan.
Hire purchase	Retail store	Retail goods, such as computers, whiteware, household goods	Set up fee Insurance may be compulsory	May have interest free deals, but interest rates will vary a lot
Credit card	Retail Bank	Any items	Annual fee Joint card holder fee	Standard cards very between 12 – 21%

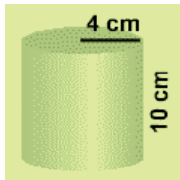
EXERCISE

1.3

1. A paper clip weighs approximately one gram, estimate how many paper clips will be in a 0.5 kilogram box of paper clips.
2. How many kg of ground chicken should you buy to make 120 burgers, if each burger will weigh 50grams before cooking?
3. Jone is recovering from surgery. The doctor told him to lift no more than 20 kg for the next 3 weeks .A carton of 48 powdered kitchen cleaner each weighs 500g each. Should Jone lift the carton?
4. A box needs to be covered in brown paper for mailing. If the box measures 3 m by 3 m by 2 m, what is the surface area of the box that will need to be covered?
The box is closed.



5. Find the surface area of this container assuming it has an open top



The bottom is closed.

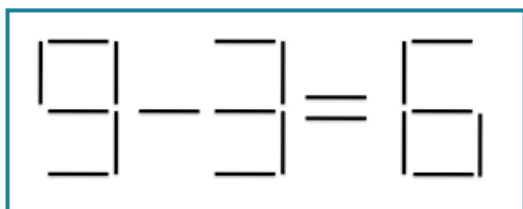
6. The cash price of a DVD player is \$280. The Hire Purchase price is \$330. If Fane pays a deposit of 15% followed by 20 equal monthly payments, find how much Fane will pay per month.
7. A computer costs \$1800 cash. For Muskaan's birthday her father decided to give the computer as a gift. The H.P. terms are: deposit 15%, followed by 36 payments of \$64.60. Find the extra paid on H.P.
8. A television set is priced at \$320 in two different shops A and B, which offer different hire purchase terms. Shop A requires 20% deposit and 12 monthly instalments of \$26.60. Shop B requires 30% deposit and 12 monthly instalments of \$23.50. Vani is planning to buy one. Which shop has the better deal for Vani and why?

9. Seru wished to buy a motorcycle at \$480. He chose to pay by hire purchase. The terms were 30% deposit with 12 monthly payments of \$33.80. How much was the total hire purchase price?
10. Coconut water from NIU Factory is sold in aluminium cans that measure 15cm in height and 3.5cm in diameter. How many cm^3 of coconut water are contained in a full can?
11. Ansaar works for an environmental protection agency which deals with land areas that have been contaminated by toxic waste. The contaminated soil covers an area of 1.62 hectares. He must remove the top 20 cm of soil in this area.
 - a. What is the area of contaminated soil in m^2 ?
 - b. What is the total volume of the contaminated soil in m^3 ?
 - c. If each truck can haul 10 m^3 of soil, how many **full** truckloads of contaminated soil will Ansaar be removing?
12. A pharmacist is filling medicine capsules. The capsules are cylinders with half spheres on each end. If the length of the cylinder is 12 mm and the radius is 2 mm, how many cubic mm of medication can one capsule hold?
13. What are the advantages of early repayment of hire purchases?
14. What are some of the hidden costs in Hire Purchases?



Test Your Knowledge

Moving Matchsticks



Think, Think,
Think..

Can you make the following equation correct by moving just one matchstick? There are two answers.

Oh, and you are not allowed to use the '**not equals to**' sign!

STRAND 2

ALGEBRA

LEARNING OUTCOMES

Sub Strand

Manipulation of Algebraic Expressions

Able to simplify by adding, subtracting, multiplying and dividing algebraic expression

- Simplify algebraic expressions
- Expanding and factorizing using various methods
- use factorizing methods like common factor ,perfect square ,difference of two squares
- factorize simple quadratic equations
- Solve equations and in equations
- Manipulate Arithmetic and geometric sequence
- Operate with matrices

HISTORY OF ALGEBRA

The word "algebra" is derived from the [Arabic](#) word *Al-Jabr*, and this comes from the treatise written in 820 by the medieval Arabian mathematician, [Muhammad ibn Mūsā al-Khwārizmī](#), entitled, in Arabic *Kitāb al-muḥtaṣar fī ḥisāb al-ğabr wa-l-muqābala*, which can be translated as *The Compendious Book on Calculation by Completion and Balancing*.

The Alexandrian mathematicians Hero of Alexandria and [Diophantus](#) continued the traditions of Egypt and Babylon, but Diophantus's book *Arithmetica* is on a much higher level and gives many surprising solutions to difficult indeterminate equations. This ancient knowledge of solutions of equations in turn found a home early in the Islamic world, where it was known as the "science of restoration and balancing." (The Arabic word for restoration, *al-jabru*, is the root of the word *algebra*.) In the 9th century, the Arab mathematician [al-Khwarizmi](#) wrote one of the first Arabic algebras, a systematic exposé of the basic theory of equations, with both examples and proofs. By the end of the 9th century, the Egyptian mathematician Abu Kamil had stated and proved the basic laws and identities of algebra and solved such complicated problems as finding x , y , and z such that $x + y + z = 10$, $x^2 + y^2 = z^2$, and $xz = y^2$. Ancient civilizations wrote out algebraic expressions using only occasional abbreviations, but by medieval times Islamic mathematicians were able to talk about arbitrarily high powers of the unknown x , and work out the basic algebra of polynomials (without yet using modern symbolism). This included the ability to multiply, divide, and find square roots of [polynomials](#) as well as knowledge of the binomial theorem. The Persian mathematician, astronomer, and poet [Omar Khayyam](#) showed how to express roots of [cubic equations](#) by line segments obtained by intersecting [conic sections](#), but he could not find a formula for the roots. A Latin translation of Al-Khwarizmi's *Algebra* appeared in the 12th century.

ALGEBRA is a method of written calculations that help us reason about numbers.

At the very outset, the student should realize that algebra is a skill. And like any skill -- driving a car, baking cakes, and playing football/rugby -- it requires practice, a lot of practice, **written practice**.

For example, the rule for adding fractions

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

The letters a and b mean: The *numbers* that are in the numerators. The letter c

means: The *number* in the denominator. The rule means:

Whatever those numbers are, add the numerators
and write their sum over the common denominator.

Algebra is telling us how to do any problem that *looks* like that. That is one reason why we use letters.

(The symbols for numbers, after all, are nothing but written marks. And so are letters! As the student will see, algebra depends only on the patterns that the symbols make.)

The numbers are the numerical symbols, while the letters are called literal symbols.

Note:

In algebra we call $a + b$ a "sum" even though we do not name an answer. As the student will see, we name something in algebra simply by how it **looks**. In fact, you will see that you do algebra with your eyes, and then what you write on the paper, follows.

What are the four operations of arithmetic, and what are their operation signs?

1. **Addition:** $a + b$. The operation sign is $+$, and is called the plus sign. Read $a + b$ as "a plus b."

For example, if a represents 3, and b represents 4, then $a + b$ represents 7.

2. **Subtraction:** $a - b$. The operation sign is $-$, and is called the minus sign. Read $a - b$ as "a minus b."

If a represents 8, for example, and b represents 2, then $a - b$ represents 6

3. **Multiplication:** $a \cdot b$. Read $a \cdot b$ as "a times b."

The multiplication sign in algebra is a centered dot. We do not use the multiplication cross $\times$, because we do not want to confuse it with the letter x .

And so if a represents 2, and b represents 5, then $a \cdot b = 2 \cdot 5 = 10$. "2 times 5 equals 10."

Do not confuse the centered dot -- $2 \cdot 5$, which means the decimal point: 2.5.

However, we often omit the multiplication dot and simply write ab . Read " a, b ." In other words, when there is no operation sign between two letters, or between a letter and a number, it always means multiplication. $2x$ means 2 times x .

4. **Division:** $\frac{a}{b}$ read as " a divided by b ".

In algebra, we use the horizontal division bar. If a represents 10, for example and b represents 2, then "10 divided by 2 is 5."

$$\frac{a}{b} = \frac{10}{2} = 5$$

What is the function of parentheses () in algebra?

Parentheses signify that we should treat what they enclose as one number.

$$3 + (4 + 5) = 3 + 9 = 12. \quad 3(4 + 5) = 3 \cdot 9 = 27.$$

When there is no operation sign between 3 and $(4 + 5)$, it means multiplication.

In algebra, we write

- a) 5 times 6? $5 \cdot 6$
- b) x times y ? xy
- c) x divide by y ? x/y
- d) x plus 5 *plus* x minus 2? $(x + 5) + (x - 2)$
- e) x plus 5 *times* x minus 2? $(x + 5)(x - 2)$

Terms versus factors.

- When numbers are added or subtracted, they are called **terms**.
- When numbers are multiplied, they are called **factors**.

Here is a sum of four terms: $a - b + c - d$.

In algebra we speak of a "sum" of terms, even though there are subtractions. In other words, anything that *looks* like what you see above, we call a sum.

Here is a **product** of four factors: $abcd$.

The word **factor** always signifies multiplication.

And again, we speak of the "product" $abcd$, even though we do not name an answer.

In the following expression, how many *terms* are there? And each term has how many factors?

$$2a + 4ab + 5a(b + c)$$

There are three terms. $2a$ is the first term. It has two factors: 2 and a .

$4ab$ is the second term. It has three factors: 4, a , and b .

And $5a(b + c)$ is all one term. It also has three factors: 5, a , and $(b + c)$.

The parentheses mean that we should treat whatever is enclosed as one number

Powers and exponents

When all the factors are equal -- $2 \cdot 2 \cdot 2 \cdot 2$ -- we call the product a power of that factor.

Thus, $a \cdot a$ is called the second power of a , or " a squared." $a \cdot a \cdot a$ is the third power of a , or " a cubed." $aaaa$ is a to the fourth power, and so on. We say that a itself is the first power of a .

Now, rather than write $aaaa$, we write a just once and place a small 4:

$$a^4 \text{ ("}a \text{ to the 4") or (}a \text{ to the fourth power)}$$

That small 4 is called an exponent. It indicates the number of times to repeat a as a factor.

8^3 ("8 to the third power" or simply "8 to the third") means $8 \cdot 8 \cdot 8$.

Name the first five powers of 2. 2, 4, 8, 16, 32. Read, then calculate each of the following.

- a) 5^2 "5 to the second power" or "5 squared" = 25.
- b) 2^3 "2 to the third power" or "2 cubed" = 8.
- c) 10^4 "10 to the fourth" = 10,000.
- d) 12^1 "12 to the first" = 12.

The student must take care not to confuse $3a$, which means

3 times a , with a^3 , which means a times a times a

$$\begin{aligned} 3a &= a + a + a, \\ a^3 &= a \cdot a \cdot a. \end{aligned}$$

What do we mean by the value of a letter?

The value of a letter is a *number*. It is the number that will replace the letter when we do the order of operations.

What does it mean to evaluate an expression?

It means to replace each letter with its value, and then do the order of operations.

Let $x = 10$, $y = 4$, $z = 2$. Evaluate the following:

In each case, copy the pattern. Copy the + signs and copy the parentheses (). When you come to x , replace it with 10. When you come to y , replace it with 4. And when you come to z , replace it with 2.

$$\begin{aligned} \text{a) } x + yz &= 10 + 4 \times 2 \\ &= 10 + 8 \\ &= 18 \end{aligned}$$

$$\begin{aligned} \text{c) } (x + y)z &= (10 + 4) \times 2 \\ &= 14 \times 2 \\ &= 28 \end{aligned}$$

$$\begin{aligned} \text{b) } x - 3(y - z) \\ &= 10 - 3(4 - 2) \\ &= 10 - 6 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{d) } (x - 3)(y - z) \\ &= (10 - 3)(4 - 2) \\ &= (7)(2) \\ &= 14 \end{aligned}$$

Expansion and factorisation of algebraic expressions

Multiplying (expanding) monomials

Multiplying by a monomial may involve:

1. Applying rules for dealing with exponents.
2. Using the distributive property.
3. Remembering rules for multiplying signed numbers.

Multiplying Signed Numbers

$$\begin{aligned} (+) \cdot (+) &= (+) \\ (+) \cdot (-) &= (-) \\ (-) \cdot (+) &= (-) \\ (-) \cdot (-) &= (+) \end{aligned}$$

Distributive Property $a(b + c) = ab + ac$

Tools for Dealing
with Monomials

Rules for Exponents

$$\begin{aligned} (xy)^n &= x^n y^n \\ (x^m)^n &= x^{mn} \\ x^m \cdot x^n &= x^{m+n} \end{aligned}$$

Example 1: monomial • monomial	$\begin{aligned}(2x^2) \cdot (4x^3) \\ &= (2 \cdot 4) \cdot (x^2 \cdot x^3) \\ &= 8 \cdot x^5 = 8x^5\end{aligned}$	Notice that the factors were regrouped and then multiplied. Also, multiply powers with the same base by adding the exponents.
Example 2: monomial • binomial	$\begin{aligned}x(x - 4) \\ &= x^2 - 4x\end{aligned}$	This problem requires the distributive property. You need to multiply each term in the parentheses by the monomial (distribute the x across the parentheses).
Example 3: monomial • trinomial	$\begin{aligned}3x(x^2 - 3x + 1) \\ &= 3x^3 - 9x^2 + 3x\end{aligned}$	Notice the distributive property at work again.
Example 4: monomial • polynomial	$\begin{aligned}3x^2(x^3 - 3x^2 - x + 5) \\ &= 3x^5 - 9x^4 - 3x^3 + 15x^2\end{aligned}$	Again, the distributive property is needed along with the rule for multiplying powers.

FACTORISING

Common Factors

The Highest Common Factor (HCF) of two (or more) monomials is the product of the greatest common factor of the numerical coefficients (the numbers out in front) and the highest power of every variable that is a factor of each monomial.

Consider: $10x^2y^3$ and $15xy^2$

The greatest common factor is $5xy^2$.

The largest factor of 10 and 15 is 5.

The highest power of x that is contained in both terms is x .

The highest power of y that is contained in both terms is y^2

When factorizing polynomials, first look for the largest monomial which is a factor of each term of the polynomial. Factor out (divide each term by) this largest monomial.

Example: 1 Factorize: $4x + 8y$

The largest integer that will divide evenly into 4 and 8 is 4. Since the terms do not contain a variable (x or y) in common, we cannot factor any variables.

The greatest common factor is 4. Divide each term by 4.

Answer: $4(x + 2y)$

Example 2: Factorize: $15x^2y^3 + 10xy^2$

The largest integer that will divide evenly into 15 and 10 is 5. The highest power of x present in both terms is the Highest power of y present in both term is y^2 . The HCF is $5xy^2$. Divide each term by the LCF.

Answer: $5xy^2 (3xy + 2)$

DIFFERENCE OF TWO SQUARES

$$a^2 - b^2$$

This expression is called a difference of Two Squares (Notice the subtraction sign between the terms a^2 and b^2 .)

The factors of $a^2 - b^2$ are $(a - b)$ and $(a + b)$

Example 1:**Factorize:** $x^2 - 9$

Both x^2 and 9 are perfect squares. Since subtraction is occurring between these squares, this expression is the difference of two squares.

These answers could also be negative values, but positive values will make our work easier

What times itself will give x^2 ? The answer is x .

What times itself will give 9? The answer is 3

The factors are $(x + 3)$ and $(x - 3)$. **Answer:** $(x + 3)(x - 3)$ or $(x - 3)(x + 3)$ (order is not important)

Example 2:**Factorize** $4y^2 - 36y^6$

There is a **common factor** of $4y^2$ that can be factored out first in this problem, to make the problem easier.

$$4y^2 (1 - 9y^4)$$

In the factor $(1 - 9y^4)$, 1 and $9y^4$ are perfect squares (their coefficients are perfect squares and their exponents are even numbers). Since subtraction is occurring between these squares, this expression is the difference of two squares. What times itself will give 1? The answer is 1.

What times itself will give $9y^4$? The answer is $3y^2$.

The factors are $(1 + 3y^2)$ and $(1 - 3y^2)$.

Answer: $4y^2 (1 + 3y^2)(1 - 3y^2)$ or $4y^2 (1 - 3y^2)(1 + 3y^2)$

If you did not see the **common factor**, you can begin with observing the perfect squares. Both $4y^2$ and $36y^6$ are perfect squares (their coefficients are perfect squares and their exponents are even numbers). Since subtraction is occurring between these squares, this expression is the difference of two squares.

Example 3: What times itself will give $4y^2$? The answer is $2y$.

What times itself will give $36y^6$? The answer is $6y^3$.

The factors are $(2y + 6y^3)$ and $(2y - 6y^3)$.

Answer: $(2y + 6y^3)(2y - 6y^3)$ or $(2y - 6y^3)(2y + 6y^3)$

These answers can be further factored as each contains a common factor of $2y$:

$$2y(1 + 3y^2) \cdot 2y(1 - 3y^2) = 4y^2(1 + 3y^2)(1 - 3y^2)$$

Be careful!!

This process of factoring does **NOT** apply to $a^2 + b^2$.

Factorizing Completely

Some polynomials cannot be factored into the product of two binomials with integer coefficients, (such as $x^2 + 16$), and are referred to as **prime**. Other polynomials contain a multitude of factors.

"**Factorizing completely**" means to continue factoring until no further factors can be found. More specifically, it means to continue factoring until all factors other than monomial factors are prime factors. You will have to look at the problems very carefully to be sure that you have found all of the possible factors.

To factor completely:

1. Search for a greatest common factor. If you find one, factor it out of the polynomial.
2. Examine what remains, looking for a trinomial or a binomial which can be factored.
3. Express the answer as the product of all of the factors you have found.

**Factorise completely:**

1. Search for the greatest common factor. In this problem, the greatest common factor is 5.

$$5x^2 - 45 = 5(x^2 - 9)$$

2. Now, examine the binomial $x^2 - 9$. (Notice how the factor of 5 is tagging along and remains as part of the answer.)

$$5(x^2 - 9) = 5(x - 3)(x + 3)$$

3. Since the binomials $(x - 3)$ and $(x + 3)$ cannot be factored further, we are done. Express the answer as the product of all of the factors.

EXERCISE

2.1

1. Evaluate each of the following expressions using each of the values of x in the list :

a. $x^2 + 2x + 3$ ($x = -2, -1, 0, 1, 2$)

b. $\frac{x + 2}{2x - 5}$ ($x = -2, -1, 0, 2, 3$)

c. $\frac{3x - 2}{x^2 - 4}$ ($x = -1, 0, 1, 2$)

2. Simplify

a. $-2ab - 8a$

b. $24kj - 16k^2j$

c. $-a^2b - b^2a$

d. $12k^2j + 24k^2j^2$

e. $72b^2q - 18b^3q^2$

f. $4(y - 3) + k(3 - y)$

g. $a^2(a - 1) - 25(a - 1)$

h. $b m(b + 4) - 6m(b + 4)j$

i. $a^2(a + 7) + 9(a + 7)$

j. $3b(b - 4) - 7(4 - b)$

k. $a^2b^2c^2 - 1$

3. Simplify the following algebraic expressions

a. $2x^2 + 3x - 7 + 2x + 4$

b. $x^2 - 2x + 2x + 4 + x^2$

c. $x^2 - x + 2x + 3 - x^2 - 5$

d. $3x^2 - 7x + 2 + 2x - 5 - 2x^2$

e. $3x^2 - 7x + 3 + 2x^2 - 5x - x^2$

f. $x^2 + 4x + 1 - 2x^2 - 2x + x^2 + 1$

g. $(5x - 2) - 2(x - 3)$

h. $(x^2 - x - 2) - (2x^2 - x - 1)$

i. $(3x^2 - x + 5) - (x^2 - 2x - 3)$

j. $2(x^2 - x - 2) - (x^2 + 3x - 4)$

k. $\frac{2x + 3}{7} + \frac{x - 4}{7}$

l. $\frac{x + 3}{x - 1} - \frac{x - 4}{x - 1}$

m. $a - \frac{ab}{b - a}$

4. Expand and simplify

a. $3(5x + 4) - 2(2x + 3)$

b. $\frac{3}{4}(4x - 8)$

5. Factorize the following

a. $5y - 30$

b. $2x^3 - 6x$

c. $4x^4 - 6x^3 + 2x^2 + 2x$

d. $3x(x + 2) - 4(x + 2)$

e. $x^2 - 25$

f. $16 - x^2$

g. $9y^2 - 49$

h. $(x - 1)^2 - 4$

i. $16y^2 - 0.09$

j. $81c^2 - 144m^2$

k. $4a^2b^2 - 49n^6$

l. $5c + 5d$

m. $g^3 - g$

n. $6x^2 - 6y^2$

o. $3x^2 - 27$

p. $y^4 - 81$

q. $25x^6 - 121y^4$

r. $3b^2 + 15b + 12$

s. $5c - 4 - 2c + 1 = 8c + 2$

6. Factorize the following

a. $6x + a + 2ax + 3$

c. $x^2 - 6x + 5x - 30$

b. $5x + 10y - ax - 2ay$

d. $a^2 - 2a - ax + 2x$

e. $5xy - 3y + 10x - 6$

f. $ab - a^2 - a + b$

7. Factorize the following

a. $5x - x^2$

b. $25 - 5x^2$

Sub Strand**Equations and In-equations**

- Able to solve linear equation
- To solve quadratic and cubic equations in factored form
- Solve simple linear in - equations and show the solution on a Cartesian

The key to solving simple algebraic equations containing a single unknown (e.g. $x + 6 = 10$) is to realize that the equation is an equality. As long as you do the same mathematical operation (e.g. add a constant, subtract a constant, multiply by a constant, and divide by

a constant) to both sides of the equation, the equality is still an equality.

This includes squaring both sides of the equation or taking the square root of both sides of the equation.

How to solve Algebraic Equations

First, let's take a look at the equation we want to solve. We want to solve for x . By this, we simply want to get the ' x ' alone on one side of the equal sign

$$5x + 10 - 5 = 30$$

The key to solving this and any equation is to use 'reverse operations.' This means to do the opposite. For example, in this equation we would want to add 5 to each side to cancel out the -5. Understand? Not too tough!!

$$5x + 10 = 35$$

Now that we have rid of the -5, let's get rid of the +10 by subtracting 10 from both sides.

$$5x = 25$$

Now that we have subtracted 10 from both sides we want to get rid of the 5 to get ' x ' by itself. The ' $5x$ ' means '5 times x '. The opposite of multiplication is division. We now want to divide both sides by 5 to get ' x ' by itself.

Example 1

To solve for x , it is necessary to subtract
6 from both sides of the equation

$$\begin{aligned}x + 6 &= 10 \\(x + 6) - 6 &= (10) - 6 \\x &= 4\end{aligned}$$

Example 2

To solve for x , you need to add 6 to
both sides of the equation and then divide both sides by 2.

$$\begin{aligned}2x - 6 &= -3 \\(2x - 6) + 6 &= (-3) + 6 \\2x &= 3 \\x &= 3/2 \text{ or } 1.5\end{aligned}$$

Example 3 To isolate x , you need to

- (1) multiply through by 6,
- (2) subtract 2 from both sides, and
- (3) divide both sides by 5.

$$\begin{aligned}\frac{5x + 2}{6} &= 2 \\6 \left[\frac{5x + 2}{6} \right] &= (6)(2) \\5x + 2 &= 12 \\(5x + 2) - 2 &= 12 - 2 \\5x &= 10 \\\frac{5x}{5} &= \frac{10}{5} \\x &= 2\end{aligned}$$

Example 4

To solve for x this time, you need to

1. multiply both sides of the equation by 4 and 3 to
cancel out the denominator in line 2.
2. use the distributive law, by adding and subtracting,

$$\begin{aligned}\frac{2x + 3}{4} &= \frac{x + 7}{3} \\(4)(3) \left[\frac{2x + 3}{4} \right] &= (4)(3) \left[\frac{x + 7}{3} \right] \\(3)(2x + 3) &= (4)(x + 7) \\6x + 9 &= 4x + 28 \\(6x - 4x) &= (28 - 9) \\x(6 - 4) &= 19 \\2x &= 19 \\x &= 19/2 = 9.5\end{aligned}$$

- move the x terms to one side, and the non- x terms to the other side in line
 - a. use the associative law to simplify to get line 7, and
 - b. divide both sides by 2.

3. This process is called "cross-multiplying." This entails multiplying the numerator of one side of the equality by the denominator of the other side of the equality. When this is done, the very same line 3 results. The rest of the problem is done identical.

$$\begin{aligned}
 \frac{2x+3}{4} &= \frac{x+7}{3} \\
 \frac{2x+3}{4} \times \frac{x+7}{3} & \\
 (3)(2x+3) &= (4)(x+7) \\
 6x+9 &= 4x+28 \\
 (6x-4x) &= (28-9) \\
 x(6-4) &= 19 \\
 2x &= 19 \\
 x &= 19/2 = 9.5
 \end{aligned}$$

Example 5

This problem could be very complicated and become a quadratic equation. However, because it has a perfect square on both sides, if you simply take the square root of both sides of the equality, you are left in line 3 with a straightforward algebra problem as you solve for the positive root

$$\begin{aligned}
 49 &= \frac{(3x+8)^2}{x^2} \\
 \sqrt{49} &= \sqrt{\frac{(3x+8)^2}{x^2}} \\
 7 &= \frac{3x+8}{x} \\
 7x &= 3x+8 \\
 7x-3x &= 8 \\
 x(7-3) &= 8 \\
 4x &= 8 \\
 x &= 2
 \end{aligned}$$

Equations involving absolute value

When you take the absolute value of a number, you always end up with a positive number (or zero). Whether the input was positive or negative (or zero), the output is always positive (or zero). For instance, $|3| = 3$, and $|-3| = 3$ also.

- **Solve $|x + 2| = 7$**

To clear the absolute-value bars, I must split the equation into its two possible two cases, one case for each sign:

$$\begin{array}{ll} (x + 2) = 7 & \text{or} \quad -(x + 2) = 7 \\ x + 2 = 7 & \text{or} \quad -x - 2 = 7 \\ x = 5 & \text{or} \quad -9 = x \end{array}$$

Then the solution is $x = -9, 5$.

- **Solve $|2x - 3| - 4 = 3$**

First, I'll isolate the absolute-value part; that is, I'll get the absolute-value expression by itself on one side of the "equals" sign, with everything else on the other side:

$$\begin{array}{ll} |2x - 3| - 4 = 3 & \\ |2x - 3| & = 7 \end{array}$$

Now I'll clear the absolute-value bars by splitting the equation into its two cases, one for each sign:

$$\begin{array}{ll} (2x - 3) = 7 & \text{or} \quad -(2x - 3) = 7 \\ 2x - 3 = 7 & \text{or} \quad -2x + 3 = 7 \\ 2x = 10 & \text{or} \quad -2x = 4 \\ x = 5 & \text{or} \quad x = -2 \end{array}$$

So the solution is $x = -2, 5$.

Remember:

Absolute value is always positive (or zero). An equation such as $|x - 2| = -6$ is never true. It has NO solution. The answer is the empty set $\emptyset$.

Solving Quadratic Equations

A *quadratic equation* is an equation containing a single variable of degree 2. Its general form is

$$ax^2 + bx + c = 0$$

where x is the variable and a , b , and c are constants ($a \neq 0$).

Example: Solve $(x - 3)(x - 4) = 0$.

If I multiply two things together and the result is zero, what can I say about those two things? I can say that at least one of them must also be zero. That is, the only way to multiply and get zero is to multiply **by** zero. (This is sometimes called "The Zero Factor Property.")

Warning: You cannot make this statement about any other number! You can only make the conclusion about the factors ("one of them must equal zero") if the product itself equals zero. You *must* always have the equation in the form "(quadratic) equals (zero)" before you can attempt to solve it. The Zero Factor Property tells me that at least one of the factors must be equal to zero.

Since at least one of the factors must be zero, I will set them *each* equal to zero:

$$x - 3 = 0 \quad \text{or} \quad x - 4 = 0$$

This gives me simple linear equations, and they're easy to solve:

$$x = 3 \quad \text{or} \quad x = 4$$

And this is the solution they're looking for: **$x = 3, 4$**

Note that " $x = 3, 4$ " means the same thing as " $x = 3$ or $x = 4$ "; the only difference is the formatting. The " $x = 3, 4$ " format is more-typically used.

Test Your Knowledge

A set of football matches is to be organized in a "round-robin" fashion, i.e., every participating team plays a match against every other team once and only once.

If a total of 45 matches were played, how many teams participated?

Solve $x^2 - 4 = 0$.

This equation is in "**(quadratic) equals (zero)**" form, so it's ready to solve. The quadratic itself is a *difference of squares*, so apply the difference-of-squares formula:

$$\begin{aligned} x^2 - 4 &= 0 \\ (x - 2)(x + 2) &= 0 \\ x - 2 &= 0 & \text{or} & x + 2 = 0 \\ x &= 2 & \text{or} & x = -2 \end{aligned}$$

Then the solution is $x = -2, 2$

Note: This solution may also be written as " $x = \pm 2$ ".

Solve $x^2 - 5x = 0$.

This two-term quadratic is easier to factor than were the previous quadratics: I can factor an x out of both terms, taking the x out front. (**Warning:** Do not "divide the x off", do not make it magically "disappear", or you'll lose one of your solutions!)

$$\begin{aligned} x(x - 5) &= 0 \\ x &= 0 & \text{or} & x - 5 = 0 \\ x &= 0 & \text{or} & x = 5 \end{aligned}$$

Then the solution to $x^2 - 5x = 0$ is $x = 0, 5$

Linear Inequalities

A Linear Inequality is like a Linear Equation (such as $y = 2x+1$) ...

... but it will have an Inequality like $<$, $>$, $\leq$, or $\geq$ instead of an “=”.

The following are examples of linear inequalities.

$$2x + 2 \leq 1$$

$$3x + 1 \geq 2$$

$$x - 6 < 7x + 2$$

The methods used to solve linear inequalities are identical to those used to solve linear equations.

The only difference occurs when there is a multiplication or a division that involves a minus sign.

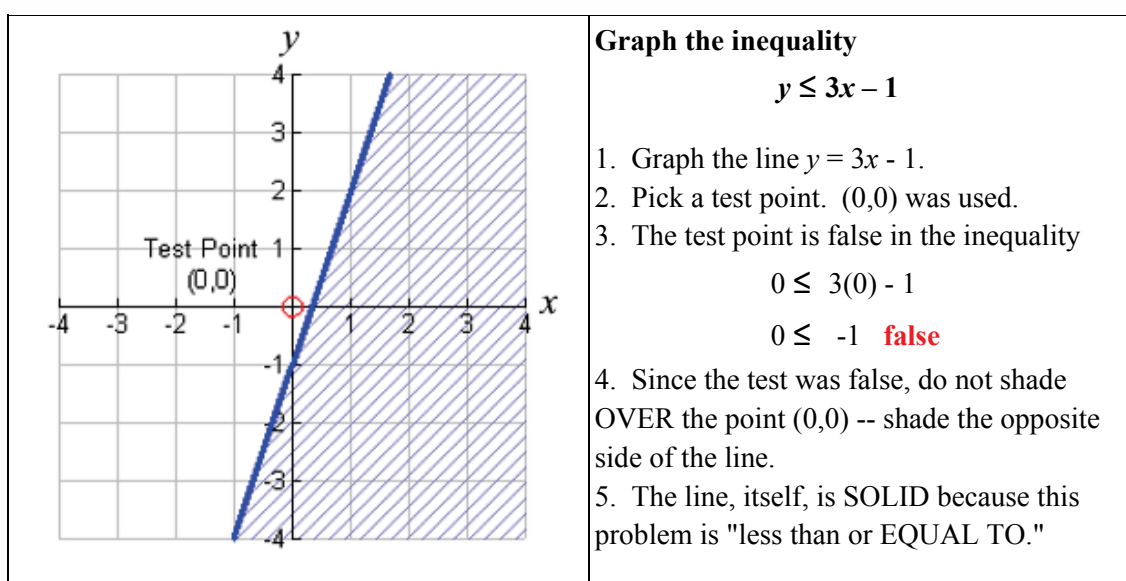
When you divide or multiply both sides of an inequality by any number with a minus sign, the direction of the inequality changes.

Graphing an inequality starts by graphing the corresponding straight line. After graphing the line, there are only two additional steps to remember.

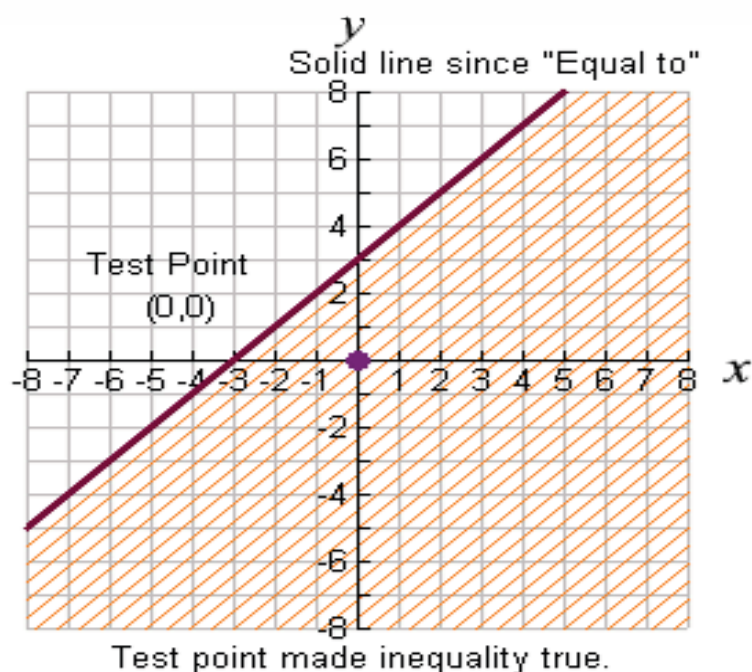
1.	Choose a point not on the line and see if it makes the inequality true. If the inequality is true, you will shade THAT side of the line -- thus shading OVER the point. If it is false, you will shade the OTHER side of the line -- not shading OVER the point.
2.	If the inequality is LESS THAN OR EQUAL TO or GREATER THAN OR EQUAL TO, the line is drawn as a solid line. If the inequality is simply LESS THAN or GREATER THAN, the line is drawn as a dashed line.

Graphing an Inequality

1. Solve the equation for **y** (if necessary).
2. Graph the equation as if it contained an = sign.
3. Draw the line **solid** if the inequality is $\leq$ or $\geq$
4. Draw the line **dashed** if the inequality is $<$ or $>$
5. Pick a point **not** on the line to use as a test point.
The point (0,0) is a good test point if it is not on the line.
6. If the point makes the inequality true, shade that side of the line. If the point does not make the inequality true, shade the opposite side of the line.



Graph the Inequality $y \leq x + 3$



EXERCISE

2.2

1. Solve the following equations

a. $2y - 3 = 7$

b. $-3w = 0$

c. $4z = 16$

d. $12t + 0 = 144$

e. $7 + 5x = 62$

f. $55 = 5y + 3$

g. $5z = 3z + 45$

h. $23a - 12 = 6 + 2a$

i. $18 - 2p = p + 2$

j. $4q = 16$

k. $3f - 10 = 10$

l. $-(-16 - r) = 13r - 1$

m. $6d - 2 + 2d = -2 + 4d + 8$

n. $3v + 16 = 4v - 10$

o. $10k + 5 + 0 = -2k + -3k + 80$

p. $8(j - 4) = 5(j - 4)$

q. $6 = 6(m + 7) + 5m$

r. $12 - 6b + 34b = 2b - 24 - 64$

s. $6c + 3c = 4 - 5(2c - 3)$

t. $(3x + 2)(3x - 4) = 0$

u. $(5a - 9)(a + 6) = 0$

v. $(2x + 3)(2x - 3) = 0$

w. Solve for x: $(2x - 3)(2x - 3) = 0$

x. Solve for x: $20x + 25x^2 = 0$

y. $(x - 2)(2x + 3)(3x - 1) = 0$

1. Solve :

(a) $5 | c - 2 | = 30$

(f) $| 3x - 1 | + 5 = 3$

(b) $| -4a + 20 | = 10$

(g) $| -7y + 21 | = 2$

(c) $| 8f | = 17$

(h) $| -c - 4 | = 18$

(d) $\left| \frac{b - 21}{5} \right| = 3$

(e) $\left| \frac{2x + 3}{5} \right| = 6$

3. Solve the following word problems

- Jack spent half of his allowance going to the movies. He washed the family car and earned 8 dollars. What is his weekly allowance if he ended with 13 dollars ?
- Oceanside Bike Rental Shop charges 16 dollars plus 9 dollars an hour for renting a bike. Benny paid 79 dollars to rent a bike. How many hours did he pay to have the bike checked out ?
- The sum of three consecutive even numbers is 132. What is the smallest of the three numbers ?
- Muskaan had 157 dollars to spend on 6 books. After buying them she had 13 dollars. How much did each book cost ?

- e. On Monday, 368 students went on a trip to the university open day. All 6 buses were filled and 8 students had to travel in cars. How many students were in each bus ?
- f. The sum of three consecutive odd numbers is 69. What is the smallest of the three numbers ?
- g. The sum of three consecutive numbers is 129. What is the smallest of the three numbers ?
- h. Ramesh sold half of his comic books and then bought 6 more. He now has 17. How many did he begin with ?
- i. Sereana bought a soft drink for 4 dollars and 7 bars of chocolate. She spent a total of 39 dollars. How much did each candy bar cost ?
- j. The cost of adding a garage to a home is 5 times as much as the cost of painting the house. If the garage addition costs \$15,000, find the cost of painting the house.
- k. The length of a rectangle is 5 times the width. If the width is 7 meters, what is the length?

4 Solve: a). $\frac{x+2}{3} = \frac{2x-4}{2}$
 b). $\frac{2(x+1)}{3} - \frac{x}{6} = 12$

5 Solve :

(a) $\frac{3x-2}{x+1} < 2$ (b) (c) $\frac{4x+4}{x-4} \leq 0$ $\frac{2x-8}{x-2} \geq 0$

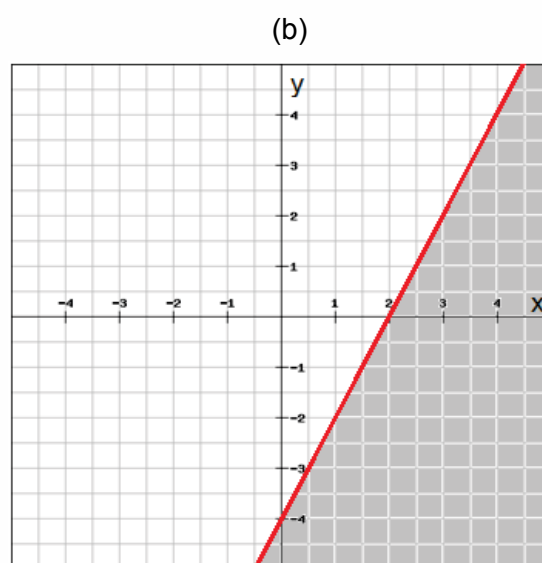
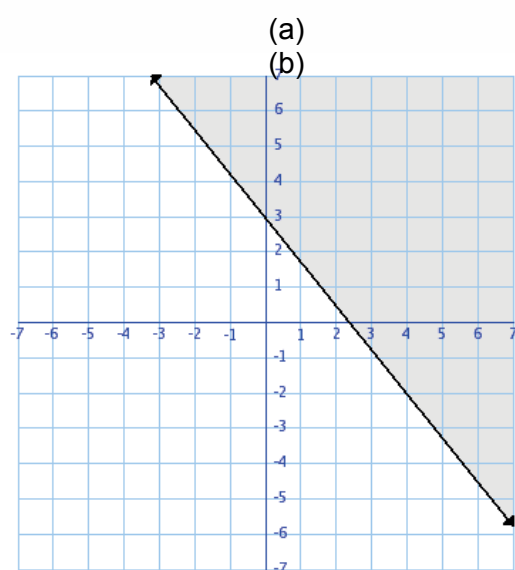
6 Solve the inequality :

(a) $-9 < 5 - 7y \leq 12$ (b) $2x + y > 4$ (c) $5x > 10$
 (d) $3x + 2 > 8$ (e) $3x - 7 < 5$ (f) $3(4 - y) \geq 9$
 (g) $-4x > -12$ (h) $-5y - 7 \leq 3$ (i) $\frac{x-2}{5} \leq -3$
 (j) $-4 \leq 3x + 2 < 5$

7. Graph the following Inequalities

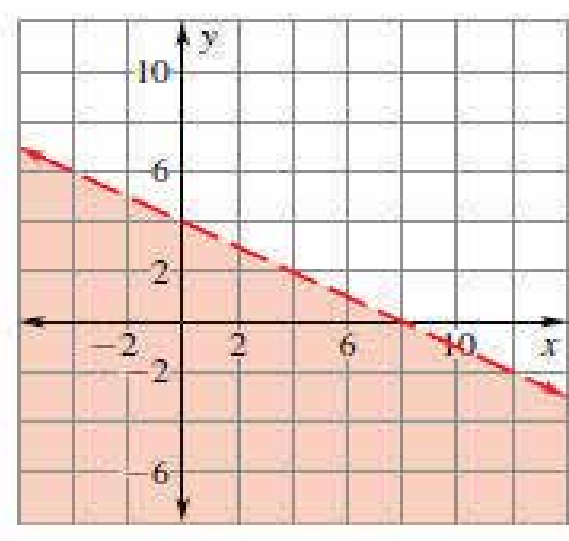
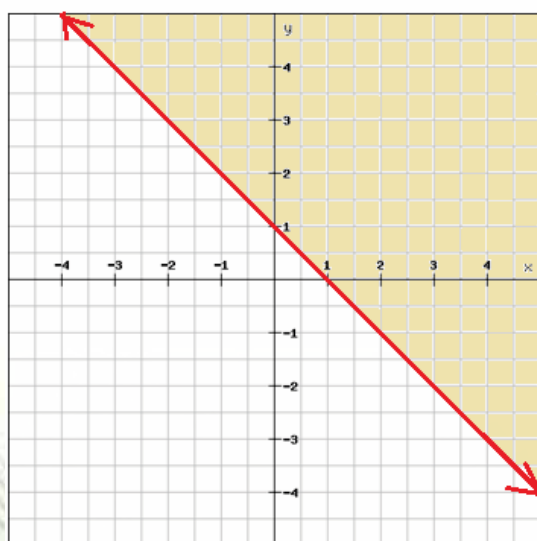
- (a) $2x + y > 4$
- (b) $2y \leq 4x + 6$
- (c) $y > 2x - 1$
- (d) $2x + 3y > 12$
- (e) $-x < y$
- (f) $-x + 3y \leq 5$

8 Describe the Inequality shown.



(c)

(d)



Sub Strand

Sequences

Sub Strand

- Understand what a sequence is
- Evaluate using Sigma Notation
- Identify simple AS and GS and use term and sum formulas.

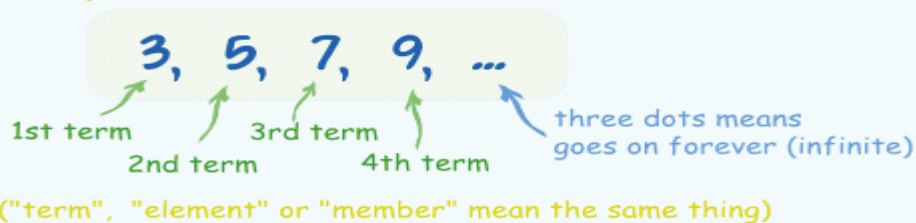
A Sequence is a set of things (usually numbers) that are in order.

While some sequences are simply random

values other sequences have a **definite pattern** that is used to arrive at the sequence's terms.

Two such sequences are the **arithmetic** and **geometric** sequences.

Sequence:



Sigma Notation

This symbol Σ (called Sigma) means "sum up".

Sum whatever appears after the Sigma: so we sum n .

But What is the Value of n ?

$$\sum n$$

The values are shown below and above the Sigma:

$$\sum_{n=1}^4 n = 1 + 2 + 3 + 4 = 10$$

It says n goes from 1 to 4, so that is **1, 2, 3 and 4**

So now we add up 1, 2, 3 and 4:

Test Your Knowledge

What's the next number in the number sequence pattern below?

1
11
21
1211
111221
312211
13112221

start at this value
go to this value

what to sum

$$\sum_{n=1}^4 n = 1+2+3+4 = 10$$

You could square n each time and sum the result:

$$\sum_{n=1}^4 n^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

You could "add up the first four terms in the sequence $2n+1$ ". The formula is easy to use ... just "plug in" the values of a , r and n .

$$\sum_{n=1}^4 2n+1 = 3 + 5 + 7 + 9 = 24$$

Arithmetic Sequence

The simplest type of numerical sequence is an arithmetic sequence.

An **arithmetic** (or linear) sequence is a sequence of numbers in which each new term is calculated by adding a constant value to the previous term.

Example 1: 1, 2, 3, 4, 5, 6, . . . is an arithmetic sequence because you add 1 to the current term to get the next term:

second term: $2 = 1 + 1$

third term: $3 = 2 + 1$

n th term: $n = (n - 1) + 1$

General Equation for the n th-term of an Arithmetic Sequence

More formally, the number we start out with is called a_1 (the first term), and the difference between each successive term is denoted d , called the common difference.

The general arithmetic sequence looks like:

$$a_1 = a_1$$

$$a_2 = a_1 + d$$

$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$$

$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$$

...

$$a_n = a_1 + d \cdot (n - 1)$$

Thus, the equation for the n th-term will be:

$$a_n = a_1 + (n - 1)d$$

Given a_1 and the common difference, d , the entire set of numbers belonging to an arithmetic sequence can be generated.

A simple test for an arithmetic sequence is to check that the difference between consecutive terms is constant:

$$a_2 - a_1 = a_3 - a_2 = a_n - a_{n-1} = d$$

To find the **sum of a certain number of terms** of an arithmetic sequence:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Here S_n is the sum of n terms (**n th partial sum**), a_1 is the first term, a_n is the n th term.

Examples

1. Find the common difference for this arithmetic sequence
5, 9, 13, 17 ...

The common difference, d , can be found by subtracting the first term from the second term, which in this problem yields **4**. Checking shows that 4 is the difference between all of the entries.

2. Find the common difference for the arithmetic sequence whose formula is
 $a_n = 6n + 32$.

The formula indicates that 6 is the value being added (with increasing multiples) as the terms increase. A listing of the terms will also show what is happening in the sequence (start with $n = 1$) 9, 15, 21, 27, 33, ...
The list shows the common difference to be 6.

3. Find the 10th term of the sequence

$$3, 5, 7, 9, \dots \quad 3. \ n = 10; \ a_1 = 3, \ d = 2$$

$$a_n = a_1 + (n-1)d$$

$$a_{10} = 3 + (10-1)2$$

$$a_{10} = 21$$

The tenth term is 21.

4. Find a_7 for an arithmetic sequence where

$$a_1 = 3x \text{ and } d = -x.$$

$$n = 7; \ a_1 = 3x, \ d = -x$$

$$a_n = a_1 + (n-1)d$$

$$a_7 = 3x + (7-1)(-x)$$

$$a_7 = 3x + 6(-x) = -3x$$

5. Find the sum of the first 12 positive even integers.

The word "sum" indicates the need for the sum formula.

positive even integers: 2, 4, 6, 8, ...

$$n = 12; \ a_1 = 2, \ d = 2$$

Notice how BOTH
formulas work
together.

We are missing a_{12} , for the sum formula, so we use the "any term" formula to find it.

Now, let's find the sum:

$$S_{12} = \frac{12(2+4)}{2} = 156$$

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_{12} &= 2 + (12-1)2 \\ a_{12} &= 24 \end{aligned}$$

There's an easier way
to solve this

Test Your Knowledge

🌈 Can you find the value of the following set:

$$1 + 2 + 3 + \dots + 200 = ??$$

Geometric Sequence

In a **Geometric Sequence** each term is found by **multiplying** the previous term by a **constant**.

Example:

2, 4, 8, 16, 32, 64, 128, 256, ...

This sequence has a factor of 2 between each number.

Each term (except the first term) is found by **multiplying** the previous term by **2**.

In General you could write a Geometric Sequence like this:

$$\{a, ar, ar^2, ar^3, \dots\}$$

where:

- **a** is the first term, and
- **r** is the factor between the terms (called the "**common ratio**")

So we would get:

$$\begin{aligned} &\{a, ar, ar^2, ar^3, \dots\} \\ &= \{1, 1 \times 2, 1 \times 2^2, 1 \times 2^3, \dots\} \\ &= \{1, 2, 4, 8, \dots\} \end{aligned}$$

Example: {1, 2, 4, 8,....}

The sequence starts at 1 and doubles each time, so:

- **a = 1** (the first term)
- **r = 2** (the "common ratio" between terms is 2)

The Rule

You can also calculate **any term** using the Rule:

$$x_n = ar^{(n-1)}$$

(We use "n-1" because ar^0 is for the 1st term)

Example :

10, 30, 90, 270, 810, 2430, ...

This sequence has a factor of 3 between each number.

The values of **a** and **r** are:

a = 10 (the first term)

r = 3 (the "common ratio")

$$x_n = 10 \times 3^{(n-1)}$$

So, the **4th** term would be:

$$x_4 = 10 \times 3^{(4-1)} = 10 \times 3^3 =$$

$$10 \times 27 = 270$$

And the **10th** term would be:

$$x_{10} = 10 \times 3^{(10-1)} = 10 \times 3^9 =$$

$$10 \times 19683 = 196830$$

A Geometric Sequence can also have **smaller and smaller** values:

Example:

4, 2, 1, 0.5, 0.25, ...

This sequence has a factor of 0.5 (a half) between each number.

Its Rule is $x_n = 4 \times (0.5)^{n-1}$

Sum of a Geometric Sequence

To find sum of a geometric sequence :

$$a + ar + ar^2 + \dots + ar^{(n-1)}$$

Each term is ar^n , where n starts at 0 and goes up to $n-1$.

Use this formula

a is the first term
r is the "**common ratio**" between terms
n is the number of terms

$$\sum_{n=0}^{n-1} (ar^n) = a \left(\frac{1 - r^n}{1 - r} \right)$$

Example: Sum the first 4 terms of

10, 30, 90, 270, 810, 2430, ...

This sequence has a factor of 3 between each number.

The values of **a**, **r** and **n** are:

- **a = 10** (the first term)
- **r = 3** (the "common ratio")
- **n = 4** (we want to sum the first 4 terms)

So:

$$\sum_{n=0}^{n-1} (ar^n) = a \left(\frac{1 - r^n}{1 - r} \right)$$

Becomes:

$$\sum_{n=0}^{4-1} (10 \cdot 3^n) = 10 \left(\frac{1 - 3^4}{1 - 3} \right) = 400$$

You could check it yourself:

$$10 + 30 + 90 + 270 = 400$$

And, yes, it was easier to just add them in this case, because there were only 4 terms. But imagine you had to sum up lots of terms, and then the formula is better to

EXERCISE

2.3

1. For each of the following sequences, determine whether it is arithmetic, geometric, or neither:

- (a) 5, 9, 13, 17, ...
- (b) 1, 2, 4, 8, ...
- (c) 1, 1, 2, 3, 5, 8, 13, 21, ...
- (d) 81, 9, 3, $\frac{1}{3}$, ...
- (e) 512, 474, 436, 398, :

2. For each of the following arithmetic sequences, find the values of a , d , and the S_n indicated.

- | | |
|-----------------------------------|--|
| a. 1, 4, 7, ... , (S_{10}) | f. 6, 8, 10, ... , (S_{12}) |
| b. 8, 6, 4, ... , (S_{12}) | g. $2, 2\frac{1}{2}, 3, \dots$, (S_{19}) |
| c. 8, 4, 0, ... , (S_{20}) | h. $6, 5\frac{3}{4}, 5\frac{1}{2}, \dots$, (S_{10}) |
| d. 20, 15, 10, ... , (S_6) | i. $7, 6\frac{1}{2}, 6, \dots$, (S_{14}) |
| e. 40, 30, 20, ... , (S_{18}) | j. 0, 5, 10, ..., (S_n) |

3. Find the first term, ratio and term indicated for each of the geometric progressions.

- | | |
|-----------------------------------|--|
| a. 1, 3, 9, ... , (S_{10}) | f. 0.005, 0.05, 0.5, ... , (S_{11}) |
| b. 4, 8, 16, ... , (S_{10}) | g. 6, 12, 24, ... , (S_6) |
| c. 18, 6, 2, ... , (S_{10}) | h. 1.4, 0.7, 0.35, ..., (S_5) |
| d. 1000, 100, 10, ... , (S_7) | i. 68, 34, 17, ... , (S_{10}) |
| e. 32, 8, 2, ... , (S_{14}) | j. $8, 2, \frac{1}{2}, \dots$, (S_{11}) |

4. Find the sum of the first 20 terms of the sequence 4, 6, 8 10, ...

5. Find the sum of the sequence -8, -5, -2, ..., 7

6. Find the 9th term of the sequence

$1, \sqrt{2}, 2, \dots$

7. Find a_6 for an arithmetic sequence where $a_1 = 3x + 1$ and $d = 2x + 6$.

8. Find t_{12} for a geometric sequence where $t_1 = 2 + 2i$ and $r = 3$

9. Find the indicated sum

$$\sum_{k=1}^{14} (1 - 2k)$$

10. How many terms of the arithmetic sequence -3, 2, 7, ... must be added together for the sum of the series to be 116?

11. A display of cans on a grocery shelf consists of 20 cans on the bottom, 18 cans in the next row, and so on in an arithmetic sequence, until the top row has 4 cans. How many cans, in total, are in the display?

12. Find the common ratio and the seventh term of the following sequence:

$$\frac{2}{9}, \frac{2}{3}, 2, 6, 18, \dots$$

Additional Activity

1. Party Problem:

At a party, everyone shook hands with everybody else. There were 66 handshakes. How many people were at the party?

A rubber ball keeps on bouncing back to $\frac{2}{3}$ of the height from which it is dropped. Can you calculate the fraction of its original height that the ball will bounce after it is dropped and it has bounced four times without any hindrance?



Sub Strand

Matrices

A Matrix is an array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix is known by the number of rows and columns (rows by columns).

The one above has 2 Rows and 3 Columns therefore it is a “**2 by 3**” matrix and written as **2 x 3**.

One is a **matrix**, or several **matrices**.

To show how many rows and columns a matrix has we often write its **order** [rows × columns].

Example: The order of this matrix is 2×3 (2 rows by 3 columns):

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

Matrix Equality

Two matrices are equal if and only if these matrices have the same dimensions and equal corresponding elements.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\begin{array}{ll} \text{If } a = e & b = f \\ c = g & d = h \end{array}$$

Addition of Matrices

To add two matrices, just add the numbers in the matching positions:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

3+4=7

- Able to identify the order of any matrix
- To add and subtract matrix of the same order
- Use scalar multiplication
- Identify that two matrices can be multiplied and multiply it.
- Find determinant and inverse of a 2 x 2

These are the calculations:

$$\begin{array}{rcl} 3 + 4 & = & 7 \quad 8 + 0 = 8 \\ 4 + 1 & = & 5 \quad 6 - 9 = -3 \end{array}$$

The two matrices must be the same size, i.e. the rows must match in size, and the columns must match in size.

Example: A matrix with **3 rows** and **5 columns** can be added to another matrix of **3 rows** and **5 columns**. But it cannot be added to a matrix with **3 rows** and **4 columns** (the columns don't match in size).

Negative

The negative of a matrix is also simple:

$$-\begin{bmatrix} 2 & -4 \\ 7 & 10 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -7 & -10 \end{bmatrix}$$

These are the calculations:

$$\begin{array}{rcl} -(2) & = & -2 \quad -(-4) = +4 \\ -(7) & = & -7 \quad -(-10) = -10 \end{array}$$

Subtraction of Matrices

To subtract two matrices, just subtract the numbers in the matching positions: Example

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$$

These are the calculations:

$$\begin{array}{rcl} 3 - 4 & = & -1 \quad 8 - 0 = 8 \\ 4 - 1 & = & 3 \quad 6 - (-9) = 15 \end{array}$$

Note: subtracting is actually defined as the addition of a negative matrix: $A + (-B)$

Multiply by a Constant

We can multiply a matrix by some value: Example :

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

These are the calculations:

$$\begin{array}{rcl} 2 \times 4 & = & 8 \quad 2 \times 0 = 0 \\ 2 \times 1 & = & 2 \quad 2 \times -9 = -18 \end{array}$$

We call the constant a **scalar**, so officially this is called "scalar multiplication".

Multiplying a Matrix by another Matrix

When we do multiplication:

- The number of **columns of the 1st matrix** must equal the number of **rows of the 2nd matrix**.
- And the result will have the same number of **rows as the 1st matrix**, and the same number of **columns as the 2nd matrix**.

$$[\$3 \ \$4 \ \$2] \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = [\$83 \ \$63 \ \$37 \ \$75]$$

$\$3 \times 13 + \$4 \times 8 + \$2 \times 6$

In this example we multiplied a 1×3 matrix by a 3×4 matrix (note the 3s are the same), and the result was a 1×4 matrix.

In General:

To multiply an $m \times n$ matrix by an $n \times p$ matrix, the **n's** must be the same, and the result is an $m \times p$ matrix.

$$m \times n \times n \times p \rightarrow m \times p$$

To multiply a matrix **by another matrix** you need to do the "dot product" of rows and columns ...

To work out the answer for the **1st row** and **1st column**:

"Dot Product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \end{bmatrix}$$

The "Dot Product" is where you **multiply matching members**, then sum up:

$$(1, 2, 3) \bullet (7, 9, 11) = 1 \times 7 + 2 \times 9 + 3 \times 11 = 58$$

We match the 1st members (1 and 7), multiply them, likewise for the 2nd members (2 and 9) and the 3rd members (3 and 11), and finally sum them up.

Here it is for the 1st row and **2nd column**:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

$$(1, 2, 3) \bullet (8, 10, 12) = 1 \times 8 + 2 \times 10 + 3 \times 12 = 64$$

We can do the same thing for the **2nd row** and **1st column**:

$$(4, 5, 6) \bullet (7, 9, 11) = 4 \times 7 + 5 \times 9 + 6 \times 11 = 139$$

And for the **2nd row** and **2nd column**:

$$(4, 5, 6) \bullet (8, 10, 12) = 4 \times 8 + 5 \times 10 + 6 \times 12 = 154$$

And we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Order of Multiplication

In arithmetic we are used to: $2 \times 5 = 5 \times 2$

(The Commutative Law of Multiplication)

But this is **not** generally true for matrices (matrix multiplication is **not commutative**):

$$\mathbf{AB} \neq \mathbf{BA}$$

When you change the order of multiplication, the answer is (usually) **different**.

Example: See how changing the order affects this multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

Identity Matrix

The "Identity Matrix" is the matrix equivalent to the number "1":

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3 x 3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1s** on the diagonal and **0s** everywhere else.
- It's symbol is the capital letter **I**.

It is a **special matrix**, because when you multiply a matrix by it, the original is *unchanged*:

$$\mathbf{A} \times \mathbf{I} = \mathbf{A}$$

$$\mathbf{I} \times \mathbf{A} = \mathbf{A}$$

Dividing

And what about division? Well we don't actually divide matrices, we do it this way:

$$\mathbf{A}/\mathbf{B} = \mathbf{A} \times (\mathbf{1}/\mathbf{B}) = \mathbf{A} \times \mathbf{B}^{-1}$$

where $\mathbf{B}^{-1}$ means the "inverse" of B.

So we do not divide, instead we **multiply by an inverse**.

Determinant of a Matrix

The **determinant** of a matrix is a **special number** that can be calculated from the matrix. It tells us things about the matrix that are useful.

The **symbol** for determinant is two vertical lines $|$ $|$ either side.

$|\mathbf{A}|$ means the *determinant* of the matrix A

Calculating the Determinant of a 2 x 2 matrix

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just a matter of basic arithmetic. Here is how:

For a 2x2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

- Blue means positive (+ad),
- Red means negative (-bc)



Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$\begin{aligned} |B| &= 4 \times 8 - 6 \times 3 \\ &= 32 - 18 \\ &= 14 \end{aligned}$$

Inverse of a Matrix

The Inverse of a Matrix is the **same idea** as the reciprocal of a number:



Reciprocal of a Number

But we do not write $\frac{1}{A}$ (because we do not divide by a Matrix!), instead we write A^{-1} for the inverse:



(In fact $\frac{1}{8}$ can also be written as 8^{-1})

And there are other similarities:

When you **multiply a number** by its **reciprocal** you get **1**

$$8 \times \left(\frac{1}{8}\right) = 1$$

When you **multiply a Matrix** by its **Inverse** you get the **Identity Matrix** (which is like "1" for Matrices):

$$A \times A^{-1} = \mathbf{I}$$

It also works when the inverse comes first: $\left(\frac{1}{8}\right) \times 8 = 1$ and $A^{-1} \times A = \mathbf{I}$.

Definition

So we have a definition of a Matrix Inverse...

The Inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = \mathbf{I}$$

Sometimes there is no Inverse at all.

2 x 2 Matrices

Well, for a 2 x 2 Matrix the Inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$



determinant

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

Let us try an example:

$$\begin{aligned} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} &= \frac{1}{4 \times 6 - 7 \times 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \end{aligned}$$

How do we know this is the right answer?

Remember it must be true that: $A \times A^{-1} = \mathbf{I}$

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{aligned} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} &= \begin{bmatrix} 4 \times 0.6 + 7 \times -0.2 & 4 \times -0.7 + 7 \times 0.4 \\ 2 \times 0.6 + 6 \times -0.2 & 2 \times -0.7 + 6 \times 0.4 \end{bmatrix} \\ &= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

We end up with the Identity Matrix! So it must be right.

It should **also** be true that: $A^{-1} \times A = \mathbf{I}$

The Inverse May Not Exist

First of all, to have an Inverse the Matrix must be "Square" (same number of rows and columns).

But also the **determinant cannot be zero** (or you would end up dividing by zero). As an example:

$$\begin{aligned} \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} &= \frac{1}{3 \times 8 - 4 \times 6} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix} \\ &= \frac{1}{24 - 24} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix} \end{aligned}$$

24 – 24 ? That equals 0, and $\frac{1}{0}$ is **undefined**.

We cannot go any further! This Matrix has no Inverse.

Such a Matrix is called "**Singular**", which only happens when the determinant is zero.

The Inverse of A is A^{-1} only when $A \times A^{-1} = A^{-1} \times A = I$

To find the Inverse of a 2x2 Matrix:

- **swap** the positions of a and d
- **put negatives** in front of b and c, and
- **divide** everything by the determinant (ad-bc).

Sometimes there is no Inverse at all.

EXERCISE

2.4

1 What is $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$? 2 What is $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & -5 \\ 3 & -2 \end{bmatrix}$?

3. What is $\begin{bmatrix} 3 & -5 & 4 \\ -1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 2 \\ -5 & -2 & 3 \end{bmatrix}$?

4. If $A = \begin{bmatrix} -3 & 1 \\ -2 & 4 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 0 & -2 \\ -2 & 4 \end{bmatrix}$ then what is $3A - 2B$?

5. **A** is a 3×2 matrix **B** is a 2×3 matrix **C** is a 2×2 matrix **D** is a 3×3 matrix

Which of the following products does not exist?

A. AB B. AC C. BD D. CD

6. If $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, then what is AB ?

7. If $X = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$, then what is XY ?

8. If $P = \begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ and $Q = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$, then what is PQ ?

9. If $A = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $AB = BA$ then what is the value of x ?

10. What is the inverse of the matrix

$$Z = \begin{bmatrix} -8 & 4 \\ 6 & -5 \end{bmatrix}?$$

11. What is the inverse of the matrix $X = \begin{bmatrix} 4 & -3 \\ -4 & 2 \end{bmatrix}$?

12. What is the inverse of the matrix

$$Y = \begin{bmatrix} -3 & 10 \\ -2 & 5 \end{bmatrix}?$$

13. What is the inverse of the matrix

$$X = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}?$$

14. What is the inverse of the matrix

$$X = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}?$$

15. Which one of the following matrices is singular?

A. $P = \begin{bmatrix} -6 & 9 \\ 2 & 3 \end{bmatrix}$

B. $Q = \begin{bmatrix} -6 & 9 \\ -2 & -3 \end{bmatrix}$

C. $R = \begin{bmatrix} 6 & 9 \\ 2 & -3 \end{bmatrix}$

D. $S = \begin{bmatrix} -6 & 9 \\ 2 & -3 \end{bmatrix}$

16. Find the value of x and y

[i]

$$\begin{pmatrix} x & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & y \\ 2 & 1 \end{pmatrix}$$

[ii]

$$\begin{pmatrix} x & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4+y & x-1 \\ x-2 & 1 \end{pmatrix}$$

Calculations like that (but using much larger matrices) help Engineers design buildings, are used in video games and computer animations to make things look 3-dimensional, and many other places.

The calculations are done by computers, but the people must understand the formulas.

STRAND 3

FUNCTIONS

LEARNING OUTCOMES

- Determine domain and range
- Find Inverse
- Identify Function or not

What is a Function?

A function relates an input to an output.

$f(x)$	" $f(x) = \dots$ " is the classic way of writing a function. And there are other ways
--------	--

Input, Relationship, Output

A function will always have 3 three main parts:

- The input
- The relationship
- The output

Some Examples of Functions

- x^2 (squaring) is a function
- $x^3 + 1$ is also a function
- Sine, Cosine and Tangent are functions used in trigonometry
- and there are lots more!

Names

First, it is useful to give a function a **name**.

The most common name is "**f**", but you can have other names like "**g**" ... or even "**your own name**" if you want.

But let's use "f":

$$f(x) = x^2$$

function name input what to output

You would say "*f of x equals x squared*"

What goes **into** the function is put inside parentheses () after the name of the function:

So $f(x)$ shows you the function is called " f ", and " x " goes **in** (*input*)

And you will often see what a function does with the input:

$f(x) = x^2$ shows you that function " f " takes " x " and squares it.

For example: with $f(x) = x^2$:

- an *input* of 4
- becomes an *output* of 16.

In fact we can write it as $f(4) = 16$.

The " x " is just a Place-Holder!

Don't get too concerned about " x ", it is just there to show you where the input goes and what happens to it. It could be anything!

So this function:

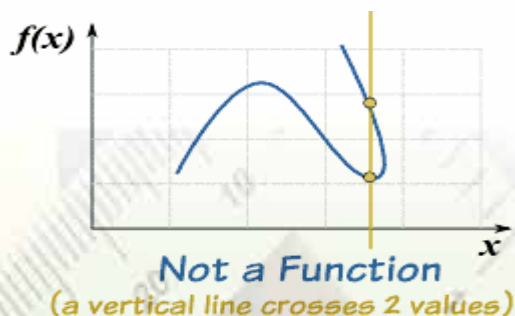
$f(x) = 1 - x + x^2$ would be the same function if it is written as :

- $f(q) = 1 - q + q^2$
- $h(A) = 1 - A + A^2$
- $w(\theta) = 1 - \theta + \theta^2$

It is just there so you know where to put the values: $f(2) = 1 - 2 + 2^2 = 3$

A function **relates** an input to an **output**.

Saying " $f(4) = 16$ " is like saying 4 is somehow related to 16. Or $4 \rightarrow 16$

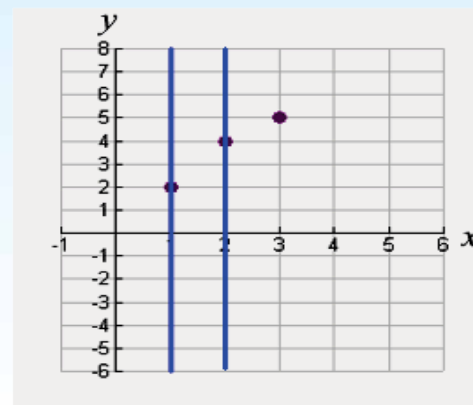


Vertical Line Test

On a graph, the idea of **single valued** means that no vertical line would ever cross more than one value.

If it **crosses more than once** it is still a valid curve, but it would **not be a function**.

The graph at the right above shows that a
Vertical line intersects only ONE point in a function.
This is called the **vertical line test** for functions



Domain and Range

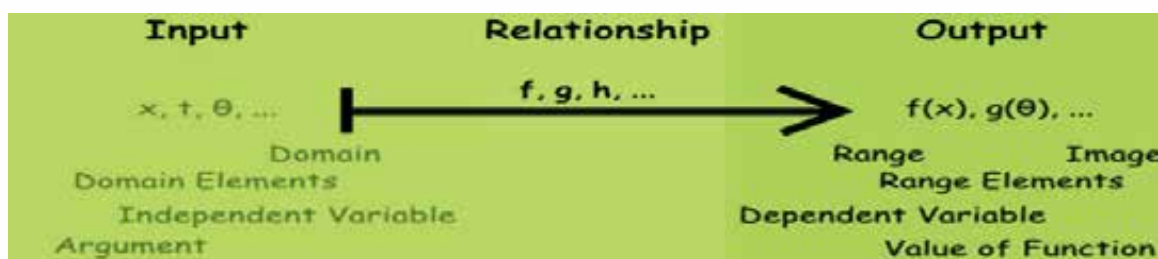
In our examples above

- the set "X" (all x-values) is called the **Domain**,
- the set of elements that get pointed to in Y (the actual values produced by the function) is called the **Range**.

So Many Names!

Functions have been used in mathematics for a very long time, and lots of different names and ways of writing functions have come about.

Here are some common terms you should get familiar with:



Example: a. with $z = 2u^3$:

- "u" could be called the "independent variable"
- "z" could be called the "dependent variable" (it **depends on** the value of u)

b. with $f(4) = 16$:

- "4" could be called the "argument"
- "16" could be called the "value of the function"

Ordered Pairs

I said I would show you many ways to think about functions, and here is another way:

You can write the input and output of a function as an "ordered pair", such as (4,16).

They are called **ordered** pairs because the input always comes first, and the output second:

(input, output)

So it looks like this:

(x , $f(x)$)

Test Your Knowledge

If $a + b = 2$ and $a^2 + b^2 = 5$
what is the value of $a^3 + b^3 = ?$

Example:

(4,16) means that the function takes in "4" and gives out "16"

Set of Ordered Pairs

A function can then be defined as a **set** of ordered pairs:

Example: $\{(2,4), (3,5), (7,3)\}$ is a function that says

"2 is related to 4", "3 is related to 5" and "7 is related 3".

Also, notice that:

- the domain is $\{2, 3, 7\}$ (the input values).
- and the range is $\{4, 5, 3\}$ (the output values).

But the function has to be **single valued**, so we also say :

"if it contains (a, b) and (a, c) , then b must equal c "

Which is just a way of saying that an input of " a " cannot produce two different results.

Example: $\{(2,4), (2,5), (7,3)\}$ is **not** a function because $\{2, 4\}$ and $\{2, 5\}$ means that 2 could be related to 4 **or** 5.

In other words it is not a function because it is **not single valued**

- A function **relates** inputs to outputs.
- A function takes elements from a set (the **domain**) and relates them to elements in a set of all the actual values y called the **range**.
- A function is a **special** type of relation where:
 - **every element** in the domain is included, and
 - any input produces **only one output** (not this **or** that)
- An input and its matching output are together called an **ordered pair**.
- So a function can also be seen as a **set of ordered pairs**.

A function and its inverse function can be described as the "DO" and the "UNDO" functions. A function takes a starting value, performs some operation on this value, and creates an output answer. The inverse function takes the output answer, performs some operation on it, and arrives back at the original function's starting value.

Finding Inverse of functions

Basically speaking, the process of finding an inverse is simply the swapping of the x and y coordinates. This newly formed inverse will be a **relation**, but may **not** necessarily be a function.

Consider this subtle difference in terminology:

Definition: INVERSE OF A FUNCTION: The relation formed when the independent variable is exchanged with the dependent variable in a given relation. *(This inverse may NOT be a function.)*

Definition: INVERSE FUNCTION: If the above mentioned inverse of a function is itself a function, it is then called an *inverse function*.

Solving for an inverse relation algebraically is a three step process:

1. Set the function = y
2. Swap the x and y variables
3. Make y the subject

Example

Find the inverse of the function

$$f(x) = x - 2$$

Remember set the function = y

$$y = x - 2$$

Swap the variables

$$x = y - 2$$

Make y the subject

$$x + 2 = y$$

$$f^{-1}(x) = x + 2$$

Finding inverse from a graph

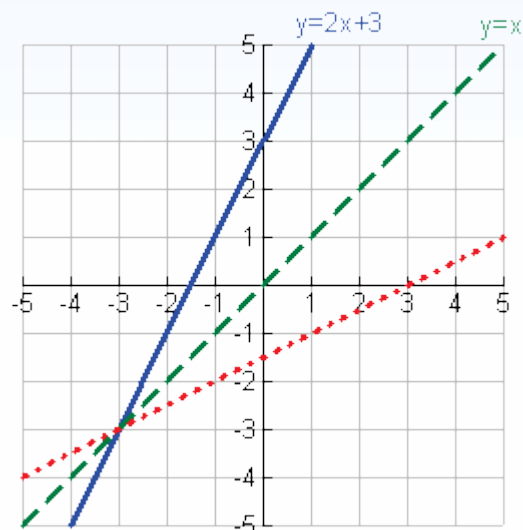
The graph of an inverse relation is the reflection of the original graph over the identity line, $y = x$. It may be necessary to restrict the domain on certain *functions* to guarantee that the inverse relation is also a *function*.

Example:

Consider the straight line, $y = 2x + 3$, as the original function. It is drawn in **blue**.

If reflected over the identity line, $y = x$, the original function becomes the **red** dotted graph. The new **red** graph is also a straight line and passes the vertical line test for functions. The inverse relation of $y = 2x + 3$ is also a function.

Not all graphs produce an inverse relation which is also a *function*.



EXERCISES

3.1

(a)

- Use the following function rule to find $f(40)$.

$$f(x) = 10 - \frac{x}{20}$$

- Use the following function rule to find $f(24)$.

$$f(x) = 5 - \frac{x}{8}$$

- Use the following function rule to find $f(3)$.

$$f(x) = 9(4)^x + 2$$

- Use the following function rule to find $f(5)$.

$$f(x) = 12(3)^x + 9$$

- Use the following function rule to find $f(r + 8)$.

$$f(n) = n + 12$$

- Given $f(x) = (x + 2)^2 - x + 3$, find:

(i) $f(2)$

(ii) $f(-2)$

(iii) $f(-4)$

(iv) $f(x + 1)$

- Is a person's weight a function of their height? Why?

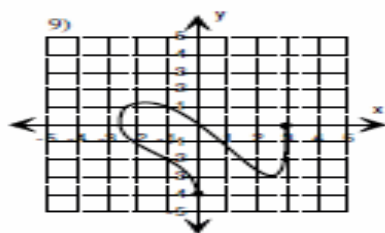
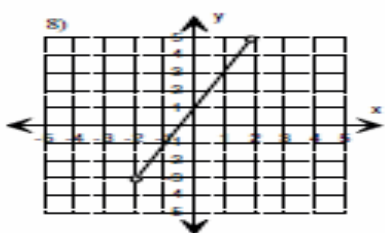
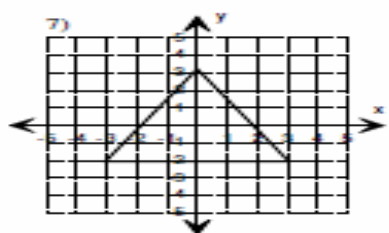
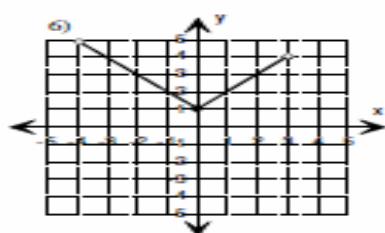
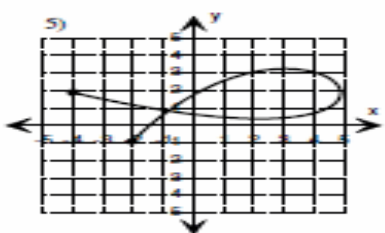
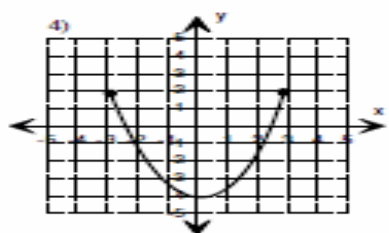
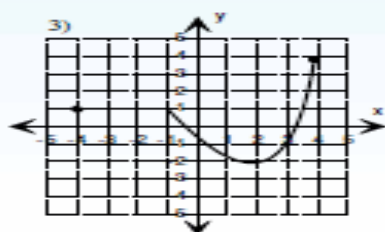
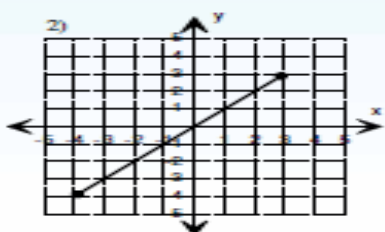
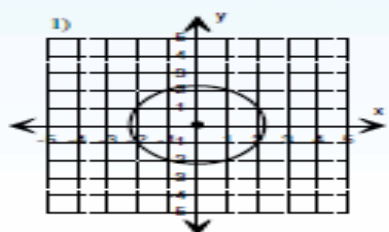
- Is the height of a rocket a function of time? Why?

- Is $y = 2x - 4$ a function .Why ?

- Is $y = x^2 - 5$ is a function .Why ?

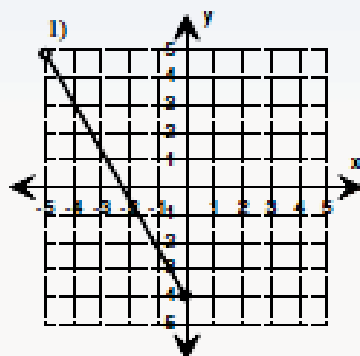
(b)

Determine whether the following is function or not



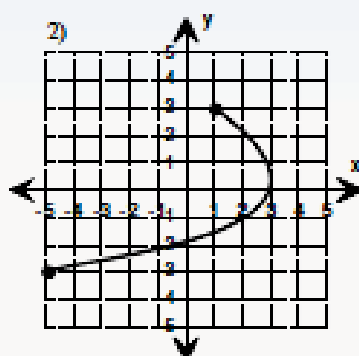
(c)

Find the Domain and Range for each graph.



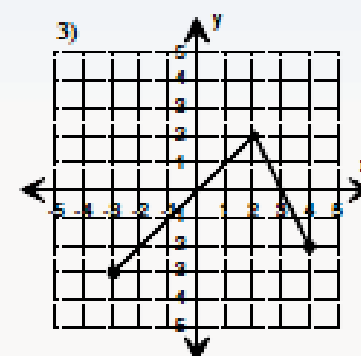
Domain : _____

Range : _____



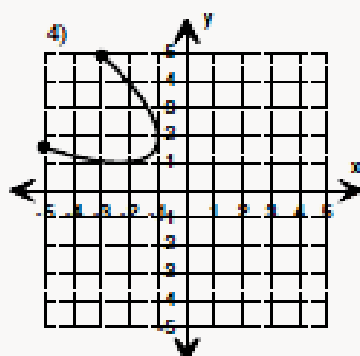
Domain : _____

Range : _____



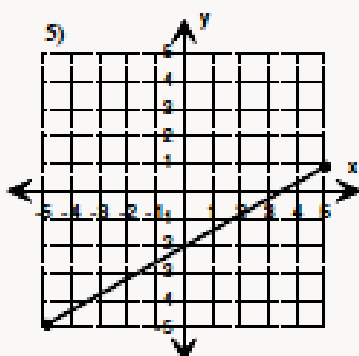
Domain : _____

Range : _____



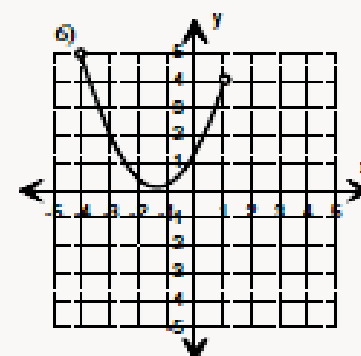
Domain : _____

Range : _____



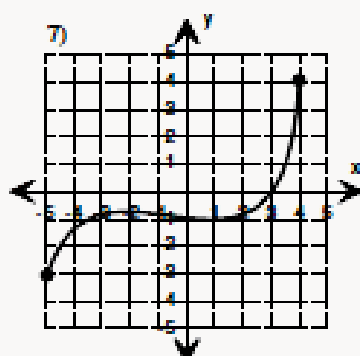
Domain : _____

Range : _____



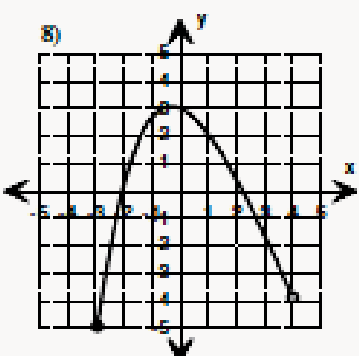
Domain : _____

Range : _____



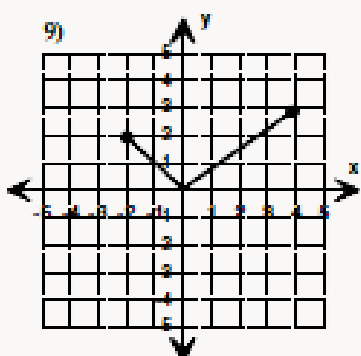
Domain : _____

Range : _____



Domain : _____

Range : _____



Domain : _____

Range : _____

STRAND 4

GRAPHS

LEARNING
OUTCOMES

Linear Functions

Linear functions are straight lines
(power of x and $y = 1$).

Form: $y = mx + c$

where m is the *gradient* and c gives the *y-intercept*

Able to draw Graphs of the following functions :

- Linear
- Quadratic
- Cubic
- Circle
- exponential



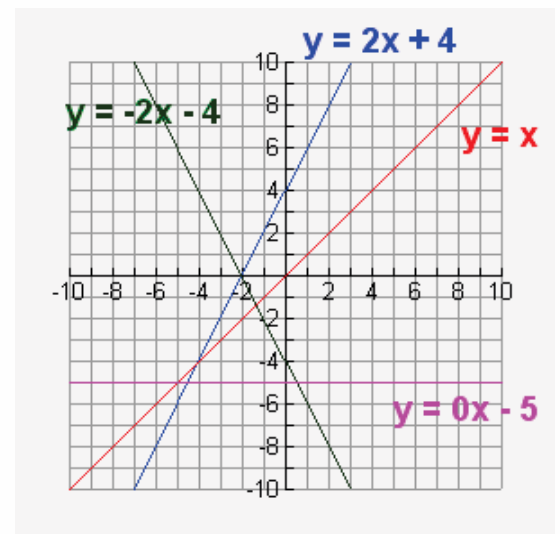
Some Observations:

1. If the slope, m , is positive, the line slants uphill. As the slope gets larger, the uphill slant of the line gets steeper. As the slope gets extremely large (a very big number), the line becomes nearly vertical. If the line is vertical, the slope is undefined (because it has no horizontal change).

2. As the slope gets smaller (closer to zero), the line loses steepness and starts to flatten. If the slope is zero, the line is horizontal (flat)

3. If the slope is negative the line slants downhill. As the slope decreases (remember -2 is > -3), the downhill slant of the line gets steeper.

4. Domain: $x \in \mathbb{R}$
Range : $y \in \mathbb{R}$



X-Intercepts and Y-Intercepts

The **x-intercept** of a line is the point at which the line crosses the x -axis. (i.e. where the y value equals 0)

$$\text{x-intercept} = (x, 0)$$

The **y-intercept** of a line is the point at which the line crosses the y -axis. (i.e. where the x value equals 0)

$$\text{y-intercept} = (0, y)$$

Example:

Find the x and y intercepts of the equation $3x + 4y = 12$

To find the x-intercept, set $y = 0$ and solve for x.

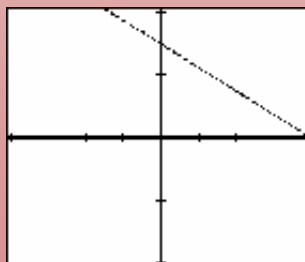
$$\begin{aligned} 3x + 4(0) &= 12 \\ 3x + 0 &= 12 \\ 3x &= 12 \\ x &= 12/3 \\ x &= 4 \end{aligned}$$

To find the y-intercept, set $x = 0$ and solve for y.

$$\begin{aligned} 3(0) + 4y &= 12 \\ 0 + 4y &= 12 \\ 4y &= 12 \\ y &= 12/4 \\ y &= 3 \end{aligned}$$

Therefore, the x-intercept is $(4, 0)$ and the y-intercept is $(0, 3)$.

The graph of the line looks like this:



Solving Equations Simultaneously

On occasions you will come across two or more unknown quantities, and two or more equations relating them. These are called simultaneous equations and when asked to solve them you must find values of the unknowns which satisfy all the given equations at the same time.

Some methods of solving equations simultaneously are the **graphical method, elimination and substitution methods**.

Graphical Method

Example . Solve simultaneously for x and y :

$$x + y = 10$$

$$x - y = 2$$

This means that we must find values of x and y that will solve both equations. We must find two numbers *whose sum is 10 and whose difference is 2*.

The two numbers, obviously, are 6 and 4:

$$6 + 4 = 10$$

$$6 - 4 = 2$$

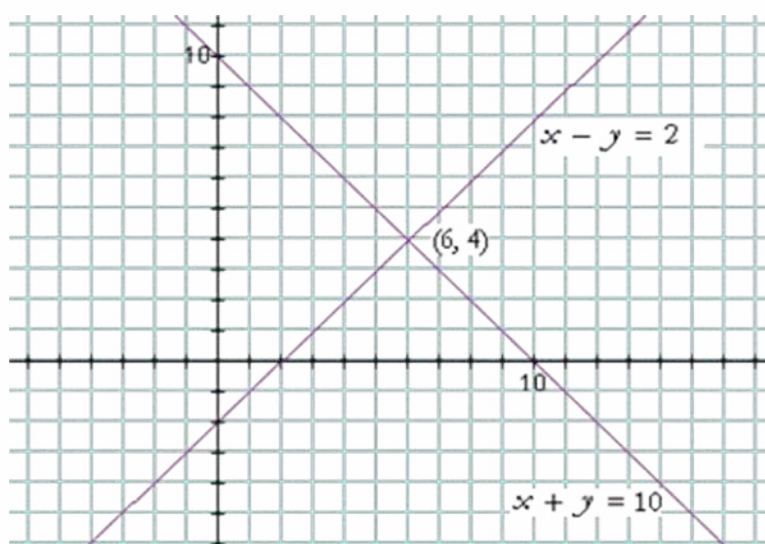
Let us represent the solution as the ordered pair (6, 4).

Now, these two equations

$$x + y = 10$$

$$x - y = 2$$

are linear equations . Hence, the graph of each one is a straight line. Here are the two graphs:



The solution to the simultaneous equations is their **point of intersection**. Why? Because that coordinate pair solves both equations. That point is the one and only point on both lines.

Solve this system of equations graphically.

$$4x - 6y = 12$$

$$2x + 2y = 6$$

Solve graphically:

To solve a system of equations graphically, graph both equations and see where they intersect. The intersection point is the solution.

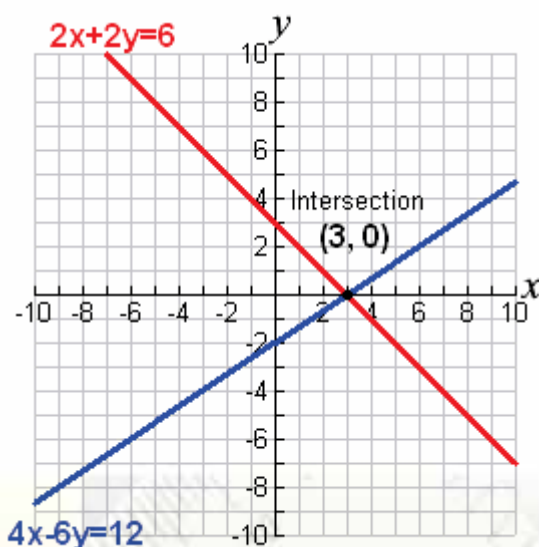
First, solve each equation for "y = ". (make **y** the subject of the formula)

$$\begin{aligned} 4x - 6y &= 12 \\ 4x &= 6y + 12 \\ 4x - 12 &= 6y \\ 6y &= 4x - 12 \\ y &= \frac{4x}{6} - \frac{12}{6} \\ y &= \frac{2}{3}x - 2 \end{aligned}$$

slope = $\frac{2}{3}$
y-intercept = -2

$$\begin{aligned} 2x + 2y &= 6 \\ 2x + 2y &= 6 \\ 2y &= -2x + 6 \\ y &= \frac{-2x}{2} + \frac{6}{2} \\ y &= -x + 3 \end{aligned}$$

slope = -1
y-intercept = 3



Graph the lines.

The slope intercept method of graphing was used in this example.

The point of intersection of the two lines, (3,0), is the solution to the system of equations.

This means that (3,0), when substituted into either equation, will make them both true. See the check.

Elimination method

Perhaps the simplest way is elimination. This is a process which involves removing or eliminating one of the unknowns to leave a single equation which involves the other unknown.

In the 'elimination' method for solving simultaneous equations, two equations are simplified by adding them or subtracting them. This eliminates one of the variables so that the other variable can be found. To add two equations, add the left hand expressions and right hand expressions separately. Similarly, to subtract two equations, subtract the left hand expressions from each other, and subtract the right hand expressions from each other. The following examples will make this clear.

Example 1: Consider these equations:

$$2x - 5y = 1$$

$$3x + 5y = 14$$

The first equation contains a '-5y' term, while the second equation contains a '+5y' term. These two terms will cancel if added together, so we will add the equations to eliminate 'y'.

To add the equations, add the left side expressions and the right side expressions separately.

$2x - 5y$	=		1
$3x + 5y$	=	+	14
$(2x - 5y) + (3x + 5y)$	=		1 + 14

Simplifying, -5y and +5y cancel out, so we have:

$$5x = 15$$

Therefore 'x' is 3.

By substituting 3 for 'x' into either of the two original equations we can find 'y'.

$$2x - 5y = -1$$

$$2(3) - 5y = -1$$

$$6 - 5y = -1$$

$$-5y = -7$$

$$y = \frac{7}{5}$$

Example 2. Solve simultaneously for x and y .

$$\begin{aligned} 2x + y &= 4 \\ x - y &= -1 \end{aligned}$$

Solution. In this case, the solution is not obvious. Here is a general strategy for solving simultaneous equations:

When one pair of coefficients are negatives of one another, add the equations vertically, and that unknown will cancel. We will then have one equation in one unknown, which we can solve.

Upon adding those equations, the y 's cancel:

$$\begin{array}{rclcl} 2x & + & y & = & 4 \\ x & - & y & = & -1 \\ \hline 3x & & & = & 3 \end{array}$$

$$x = \frac{3}{3}$$

$$x = 1.$$

To solve for y , the other unknown:

Substitute the value of x in one of the original equations.

Upon substituting $x = 1$ in the top equation:

$$2x + y = 4$$

$$2(1) + y = 4$$

$$2 + y = 4 \quad y = 2$$

If we report the solution as an ordered pair, then the solution is $(1, 2)$. Those are the coordinates of the point of intersection of the two lines.

Example 3:

The elimination method will only work if you can eliminate one of the variables by adding or subtracting the equations as in example 1 above. But for many simultaneous equations, this is not the case.

For example, consider these equations:

$$2x + 3y = 4$$

$$x - 2y = -5$$

Adding or subtracting these equations will not cancel out the 'x' or 'y' terms.

Before using the elimination method you may have to multiply every term of one or both of the equations by some number so that equal terms can be eliminated. We could eliminate 'x' for this example if the second equation had a '2x' term instead of an 'x' term. By multiplying every term in the second equation by 2, the 'x' term will become '2x', like this:

$$x \times 2 - 2y \times 2 = -5 \times 2$$

giving

$$2x - 4y = -10$$

Now the 'x' term in each equation is the same, and the equations can be subtracted to eliminate 'x':

$$\begin{array}{rcl} 2x + 3y & = & 4 \\ -2x - 4y & = & -10 \\ \hline (2x + 3y) - (2x - 4y) & = & 4 - -10 \end{array}$$

Removing the brackets and simplifying, the '2x' terms cancel out, so we have:

$$7y = 14$$

So

$$y = 2$$

The other variable, 'x', can now be found by substituting 2 for 'y' into either of the original equations.

$$\begin{array}{l} x - 2y = -5 \\ x - 2(2) = -5 \\ x - 4 = -5 \\ x = 9 \end{array}$$

Substitution Method

Here is the method of substitution:

- Solve one of the equations for one unknown in terms of the other.
- Then, substitute that in the other equation.
- That will yield one equation in one unknown, which we can then solve.

Consider example 2 from elimination:

$$\begin{array}{rcl} 2x + y & = & 4 \quad \text{-----} \rightarrow \textcircled{1} \\ x - y & = & -1 \quad \text{-----} \rightarrow \textcircled{2} \end{array}$$

Let us **solve** equation $\textcircled{1}$ for y :

$$\textcircled{1} \quad y = 4 - 2x$$

And now, **substitute** this for y in equation $\textcircled{2}$:

$$\textcircled{2} \quad x - (4 - 2x) = -1$$

This equation has only the unknown x :

$$x - 4 + 2x = -1$$

$$3x = -1 + 4$$

$$3x = 3$$

$$x = 1$$

To find y , substitute $x = 1$ in line 1):

$$y = 4 - 2 \cdot 1$$

$$y = 2.$$

EXERCISE 4.1

1. Use the method of substitution to solve each other of the pair of simultaneous equations:

- | | | | |
|---------------------|----------------|-------------------------|----------------|
| (a) $x + y = 15$, | $x - y = 3$ | (b) $x + y = 0$, | $x - y = 2$ |
| (c) $2x - y = 3$, | $4x + y = 3$ | (d) $2x - 9y = 9$, | $5x + 2y = 27$ |
| (e) $x + 4y = -4$, | $3y - 5x = -1$ | (f) $2x - 3y = 2$, | $x + 2y = 8$ |
| (g) $x + y = 7$, | $2x - 3y = 9$ | (h) $11y + 15x = -23$, | $7y - 2x = 20$ |
| (i) $5x - 6y = 2$, | $6x - 5y = 9$ | | |

2. Solve each other pair of equation given below using elimination method:

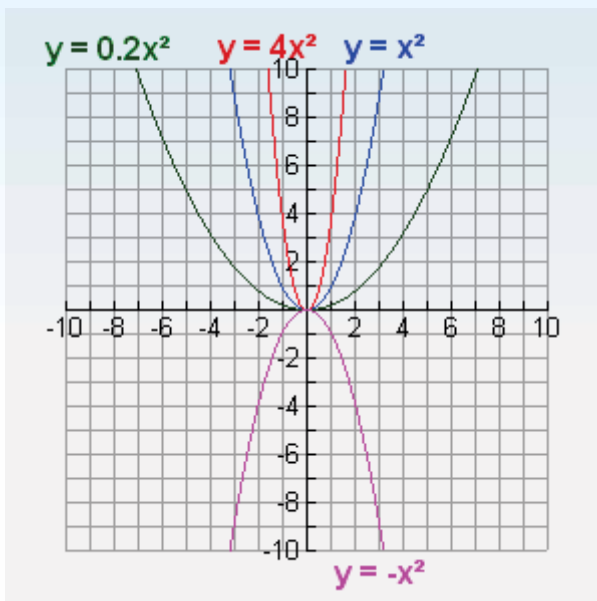
- | | | | |
|--|---|-------------------------------|------------------------|
| (a) $x + 2y = -4$, | $3x - 5y = -1$ | (b) $4x + 9y = 5$, | $-5x + 3y = 8$ |
| (c) $9x - 6y = 12$, | $4x + 6y = 14$ | (d) $2y - \frac{3}{x} = 12$, | $5y + \frac{7}{x} = 1$ |
| (e) $\frac{3}{x} + \frac{2}{y} = \frac{9}{xy}$, | $\frac{9}{x} + \frac{4}{y} = \frac{21}{xy}$ | | |
| (f) $\frac{4}{y} + \frac{3}{x} = 8$, | $\frac{6}{y} + \frac{5}{x} = 13$ | (g) $5x + \frac{4}{y} = 7$, | $4x + \frac{3}{y} = 5$ |
| (h) $x + y = 3$, | $-3x + 2y = 1$ | (i) $-3x + 2y = 5$, | $4x + 5y = 2$ |

3. Solve the following simultaneous equations:

- | | |
|--------------------------------|------------------------------------|
| a. $3a + 4b = 43$ | $-2a + 3b = 11$ |
| b. $4x - 3y = 23$ | $3x + 4y = 11$ |
| c. $5x + (4/y) = 7$ | $4x + (3/y) = 5$ |
| d. $4/(p - 3) + 6/(q - 4) = 5$ | $5/(p - 3) - 3/(q - 4) = 1$ |
| e. $(l/6) - (m/15) = 4$ | $(l/3) - (m/12) = 19/4$ |
| f. $3x + 2y = 8$ | $4x + y = 9$ |
| g. $x - y = -1$ | $2y + 3x = 12$ |
| h. $(3y/2) - (5x/3) = -2$ | $(y/3) + (x/3) = 13/6$ |
| i. $x - y = 3$ | $(x/3) + (y/2) = 6$ |
| j. $(2x/3) + (y/2) = -1$ | $(-x/3) + y = 3$ |
| k. $5x + 8y = 9$ | $2x + 3y = 4$ |
| l. $3 - 2(3a - 4b) = -59$ | $(a - 3)/4 - (b - 4)/5 = 2^{1/10}$ |

4. If $2x + 5y = 21$ and $x + y = 3$, then what is the value of x ?
5. If $7x + 2y = 3$ and $x - 3y = 30$, what is the value of y ?
6. If $2x + 5y = -1$ and $3x - 2y = 27$, then what is the value of x ?
7. If $4x - 3y = -18$ and $6x + 7y = -4$, then what is the value of y ?
8. $2x - 3y = 24$ and $3x + 4y = 2$, what is the value of $x - y$?
9. $a + b = 5$ and $3a + 2b = 20$, then what is $3a + b$ equal to?
10. If $3p - 2q = 4$ and $7p - 3q = 1$, then what is $4p - 5q$ equal to?
11. Sahid has 20 balls of four colors: yellow, green, blue, and black. 17 of them are not green, 5 are black, and 12 are not yellow. How many blue balls does Sahid have?
12. Ansaar bought 20 shirts at a total cost of R 980. If the large shirts cost R 50 and the small shirts cost R 40. How many of each size did he buy?
13. The diagonal of a rectangle is 25 cm more than its width. The length of the rectangle is 17 cm more than its width. What are the dimensions of the rectangle?
14. The sum of 27 and 12 is equal to 73 more than an unknown number. Find the number.
15. The sum of two consecutive odd numbers is 20 and their difference is 2. Find the two numbers.

Quadratic Functions



Quadratic functions are parabolas.
Form $y = ax^2$.

Some Observations:

1. If the coefficient of x^2 gets larger, the parabola becomes thinner (narrower), closer to its line of symmetry.
2. If the coefficient of x^2 gets smaller, the parabola becomes thicker (wider), further from its line of symmetry.
3. If the coefficient of x^2 is negative, the parabola opens downward

Quadratic in the form $y = x^2 - 4x$

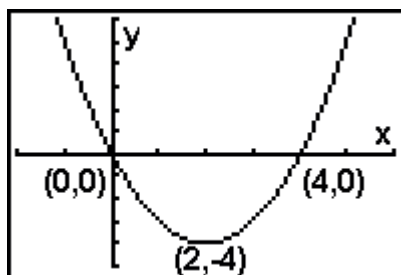
(Remember: $[-1, 5]$ means from $x = -1$ to $x = 5$ inclusive.)

In this example, the interval used for preparing the chart was given in the question. Consequently, the turning point of the parabola fell within the interval. If the question had NOT told us the interval,

how would we have known which values to place in the chart to ensure that we would see the turning point of the parabola? To guarantee that the points you choose for your chart will show the turning point, start by determining the axis of symmetry of the parabola.

x	$x^2 - 4x$	y
-1	$(-1)^2 - 4(-1)$	5
0	$(0)^2 - 4(0)$	0
1	$(1)^2 - 4(1)$	-3
2	$(2)^2 - 4(2)$	-4
3	$(3)^2 - 4(3)$	-3
4	$(4)^2 - 4(4)$	0
5	$(5)^2 - 4(5)$	5

Plot the points generated in the chart. Draw a smooth curve through the points.



The points where the graph crosses the x -axis are called the roots of $0 = x^2 - 4x$.

This parabola crosses the x -axis at $(0, 0)$ and $(4, 0)$.

Completing the Square

"Completing the Square" is where we ..

take a Quadratic Equation like this:

$$ax^2 + bx + c = 0$$

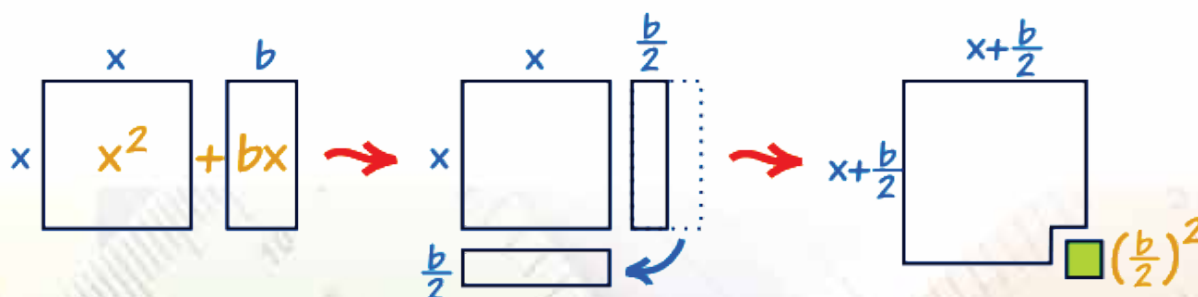


and turn it into this:

$$a(x+d)^2 + e = 0$$

Say we have a simple expression like $x^2 + bx$. Having x twice in the same expression can make life hard. What can we do?

Well, with a little inspiration from Geometry we can convert it, like this:



As you can see $x^2 + bx$ can be rearranged **nearily** into a square ...

... and we can **complete the square** with $(b/2)^2$

In Algebra, it looks like this:

$$x^2 + bx + \boxed{} + (b/2)^2 = (x+b/2)^2$$

"Complete the Square"

So, by adding $(b/2)^2$ we can complete the square.

And $(x+b/2)^2$ has x only **once**, which is easier to use.

Keeping the Balance

Now ... we can't just **add** $(b/2)^2$ without also **subtracting** it too! Otherwise the whole value changes.

So let's see how to do it properly with an example:

Start with: $x^2 + 6x + 7$

("b" is 6 in this case)

Complete the Square:

$$x^2 + 6x + \boxed{} + 7 \boxed{}$$

$$+ \left(\frac{6}{2}\right)^2 \quad - \left(\frac{6}{2}\right)^2$$

Also **subtract** the new term

Simplify it and we are done.

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 + 7 - \left(\frac{6}{2}\right)^2$$

$$\underbrace{\phantom{x^2 + 6x + \left(\frac{6}{2}\right)^2}}_{\left(x + \frac{6}{2}\right)^2} + \underbrace{\phantom{7 - \left(\frac{6}{2}\right)^2}}_{7 - 9} = (x + 3)^2 - 2$$

The result:

$$x^2 + 6x + 7 = (x + 3)^2 - 2$$

A Shortcut Approach

Expanding $(x + b)^2$ gets us $x^2 + 2bx + b^2$, and we have turned the question into this:

$$x^2 + 6x + 7 \rightarrow x^2 + 2bx + b^2 + c$$

The diagram shows the expansion of $(x + b)^2 + c$ to $x^2 + 2bx + b^2 + c$. A blue arrow points from the $6x$ term in the original expression to the $2bx$ term in the expanded form, indicating that $2b = 6$ and thus $b = 3$. An orange arrow points from the 7 term to the $b^2 + c$ term, indicating that $b^2 + c = 7$. Since $b = 3$, $9 + c = 7$, so $c = -2$.

Now we can "force" an answer:

- We know that $6x$ must end up as $2bx$, so **b must be 3**
- Next we see that 7 must become $b^2 + c = 9 + c$, so **c must be -2**

And we get the same result $(x+3)^2 - 2$ as above!

A Simple Approach

$$ax^2 + bx + c = 0 \rightarrow a(x+d)^2 + e = 0$$

$$d = \frac{b}{2a} \quad e = c - \frac{b^2}{4a} \quad \text{and}$$

The vertex form of a quadratic function is given by

$$f(x) = a(x - h)^2 + k$$

where (h, k) is the vertex of the parabola.

When written in "vertex form":

- (h, k) is the vertex of the parabola, and $x = h$ is the axis of symmetry.
- the h represents a horizontal shift (how far left, or right, the graph has shifted from $x = 0$).
- the k represents a vertical shift (how far up, or down, the graph has shifted from $y = 0$).
- notice that the h value is subtracted in this form, and that the k value is added.
If the equation is $y = 2(x - 1)^2 + 5$, the value of h is 1, and k is 5.
If the equation is $y = 3(x + 4)^2 - 6$, the value of h is -4 , and k is -6 .

Sketch the graph of $y = 3(x - 2)^2 - 4$

1. Start with the function in vertex form:

$$y = 3(x - 2)^2 - 4$$

2. Pull out the values for h and k .

If necessary, rewrite the function so you can clearly see the h and k values.

(h, k) is the vertex of the parabola.

Plot the vertex

$$y = 3(x - 2)^2 + (-4)$$

$$h = 2; \quad k = -4$$

Vertex: $(2, -4)$

3. The line $x = h$ is the axis of symmetry.

Draw the axis of symmetry.

$x = 2$ is the axis of symmetry

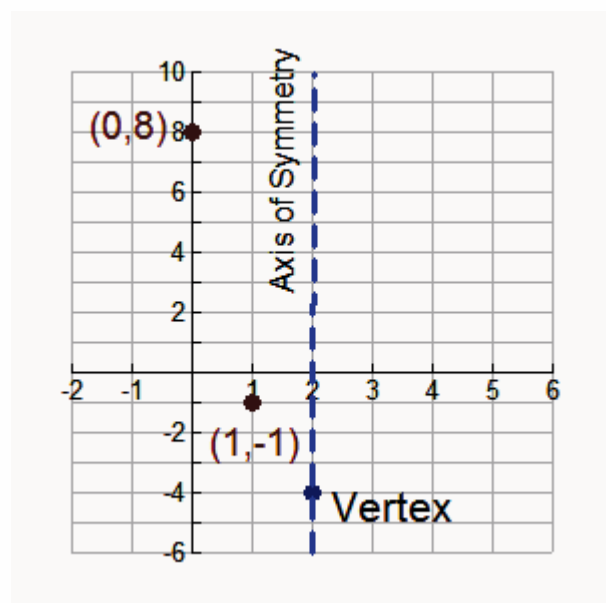
4. Find two or three points on one side of the axis of symmetry, by substituting your chosen x -values into the equation.

For this problem, we chose (to the left of the axis of symmetry):

$$x = 1; \quad y = 3(1 - 2)^2 - 4 = -1$$

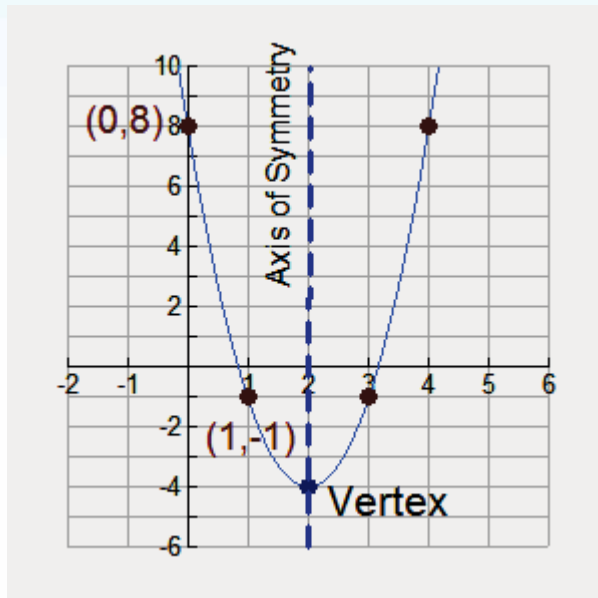
$$x = 0; \quad y = 3(0 - 2)^2 - 4 = 8$$

Plot $(1, -1)$ and $(0, 8)$.



- Plot the mirror images of these points across the axis of symmetry, or plot new points on the right side. Draw the parabola.

Remember, when drawing the parabola to avoid "connecting the dots" with straight line segments. A parabola is curved, not straight, as its slope is not constant.



Graphing Cubic Functions

A cubic function has the form

$$f(x) = ax^3 + bx^2 + cx + d - \text{expanded form}$$

$$f(x) = (x - a)(x - b)(x - c) - \text{Factorised form}$$

where **a**, **b**, **c** and **d** are real numbers and **a** is **not equal** to 0.

The domain of this function is the set of all real numbers. The range of f is the set of all real numbers. The **y intercept** of the graph of f is given by $f(0) = d$. The **x intercept(s)** are found by solving the equation

$$ax^3 + bx^2 + cx + d = 0$$

The left hand side behaviour of the graph of the cubic function is as follows:

- If the leading coefficient is positive, as x increases the graph of f is up and as x decreases indefinitely the graph of f is down.
- If the leading coefficient is negative, as x increases the graph of f is down and as x decreases indefinitely the graph of f is up.

Example 1: The basic cubic function given by

$$f(x) = x^3$$

- Find the x and y intercepts of the graph of f .
- Find the domain and range of f .
- Sketch the graph of f .

Solution to Example 1

a - The y intercept is given by
 $(0, f(0)) = (0, 0)$

- The x coordinates of the x intercepts are the solutions to $x^3 = 0$
- The x intercept are at the points $(0, 0)$.

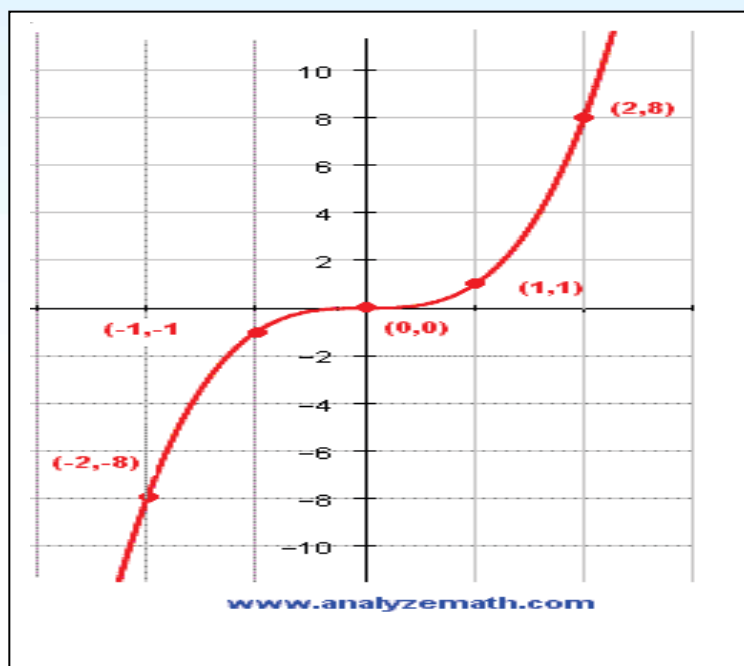
b - The domain of $f(x)$ is the set of all real numbers.

- Since the leading coefficient (of x^3) is positive, the graph of f is up on the right and down on the left and hence the range of f is the set of all real numbers.

c - Make a table of values and graph.

x	-2	-1	0	1	2
$f(x) = x^3$	-8	-1	0	1	8

- Also since $f(-x) = -f(x)$, function f is odd and its graph is symmetric with respect to the origin $(0,0)$.



Example 2: f is a cubic function given by

$$f(x) = -(x - 2)^3$$

- Find y intercepts of the graph of f .
- Find all zeros of f and their multiplicity.
- Find the domain and range of f .
- Use the y intercept, x intercepts and other properties of the graph of to sketch the graph of f .

Solution to Example 2

a - The y intercept is given by

$$(0, f(0)) = (0, 8)$$

b - The zeros of f are solutions to

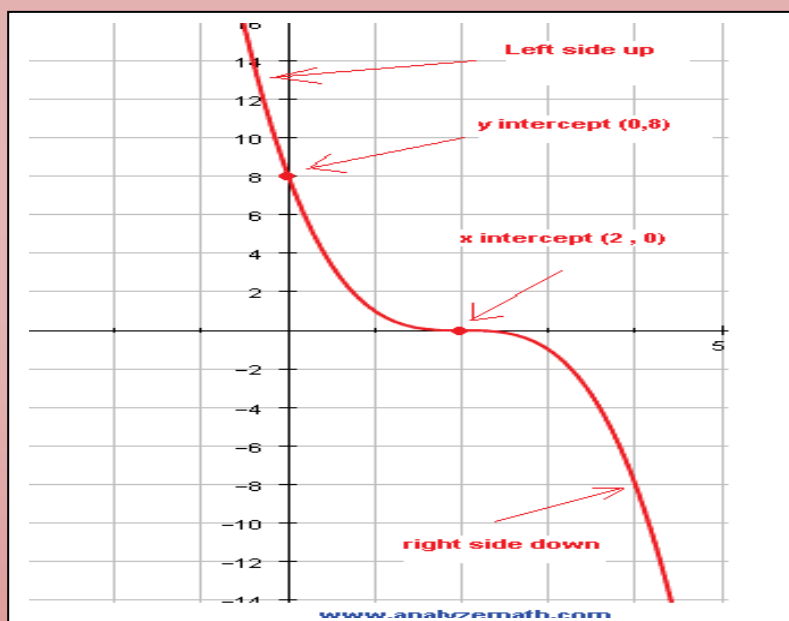
$$-(x - 2)^3 = 0$$

- Function f has one zero at $x = 2$ of multiplicity 3 and therefore the graph of f cuts the x axis at $x = 2$.
- c - The domain of $f(x)$ is the set of all real numbers.

- After expansion of $f(x)$, we can see that the leading coefficient (of x^3) is negative, the graph of f is down on the right and up on the left and hence the range of f is the set of all real numbers.

d - Properties and graph.

- At $x = 2$, the graph cuts the x axis. The y intercept is a point on the graph of f . Also the graph of $f(x) = -(x - 2)^3$ is that of $f(x) = x^3$ shifted 2 units to the right.



- The term $(x - 2)$ and reflected on the x axis because of the negative sign in $f(x) = -(x - 2)^3$. Adding to all these properties the left and right hand behaviour of the graph of f , we have the following graph.

Example 3: f is a cubic function given by

$$f(x) = (x+2)(x-1)(x+1)$$

- Factor $f(x)$.
- Find all zeros of f and their multiplicity.
- Find the domain and range of f .
- Use the y intercept, x intercepts and other properties of the graph of f to sketch the graph of f .

Solution to Example 3

a - $f(x) = x^3 + 2x^2 - x - 2 = x^2(x + 2) - (x + 2) = (x + 2)(x^2 - 1)$

b - The zeros of f are solutions to $(x + 2)(x^2 - 1) = 0$

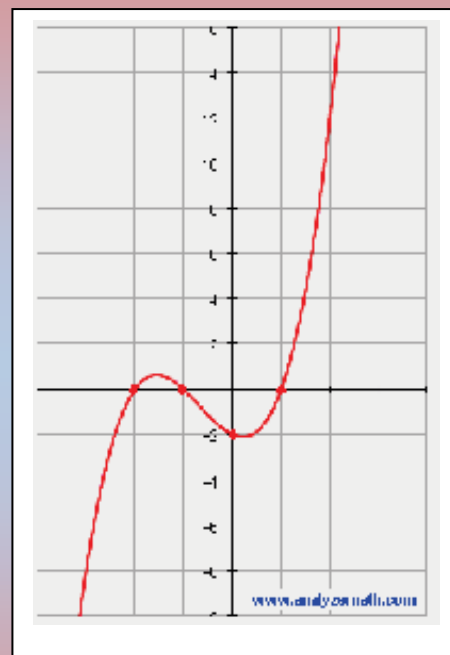
- Function f has zeros at $x = -2$, at $x = 1$ and $x = -1$. Therefore the graph of f cuts the x axis at all these x intercepts.

c - The domain of $f(x)$ is the set of all real numbers.

- The leading coefficient of $f(x)$ is positive, the graph of f is down on the left and up on the right and hence
- The range of f is the set of all real numbers.

d - Properties and graph.

- The y intercept of the graph of f is at $(0, -2)$. The graph cuts the x axis at $x = -2, -1$ and 1 . Adding to all these properties the left and right hand behaviour of the graph of f , we have the graph shown.



Example 4: f is a cubic function given by

$$f(x) = -(x - 2)(x + 1)^2$$

- Find all zeros of f and their multiplicity.
- Find the domain and range of f .
- Use the y intercept, x intercepts and other properties of the graph to sketch the graph of f .

Solution to Example 4

a. The zeros of f are solutions to

$$-(x - 2)(x + 1)^2 = 0$$

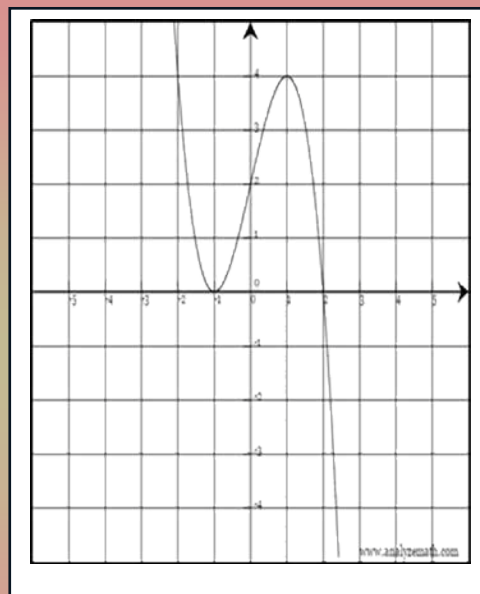
- Function f has zeros at $x = 2$ and $x = -1$ with multiplicity 2. Therefore the graph cuts the x axis at $x = 2$ and is tangent to the x axis at $x = -1$ because the multiplicity of this zero is even.

b. The domain of $f(x)$ is the set of all real numbers.

- The leading coefficient $f(x)$ is negative, the graph of f is up on the left and down on the right and hence the range of f is the set of all real numbers.

c. Properties and graph.

The y intercept of the graph of f is at $(0, 2)$. The graph cuts the x axis at $x = 2$ and is tangent to it at $x = -1$. Adding to all these properties the left and right hand behaviour of the graph of f , we have the following graph.



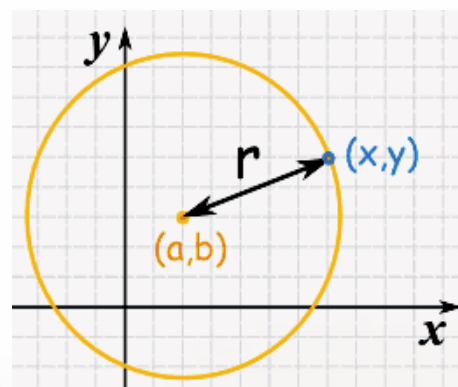
Circle

The definition of a circle is

the set of all points on a plane that are a fixed distance from a **centre**.

Let us put that center at (a, b) .

So the circle is **all the points (x, y)** that are " r " away from the centre (a, b) .



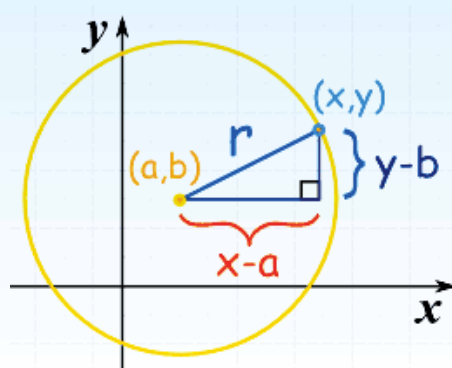
Now we can work out **exactly** where all those points are:

We make a right-angled triangle (as shown),

and then use Pythagoras ($a^2 + b^2 = c^2$):

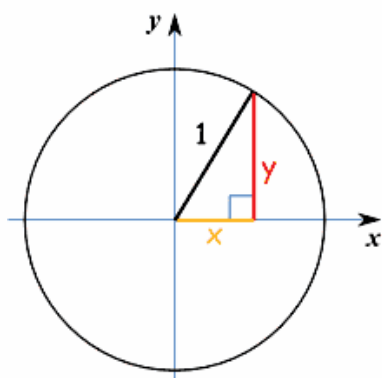
And that is the "**Standard Form**" for the equation of a circle!

$$(x - a)^2 + (y - b)^2 = r^2$$



Unit Circle

If we place the circle center at (0,0) and set the radius to 1 we get:



$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x - 0)^2 + (y - 0)^2 = 1^2$$

$$x^2 + y^2 = 1$$

which is the equation of the Unit Circle.

How to Plot a Circle by Hand

1. Plot the centre (**a, b**)
2. Plot 4 points "radius" away from the centre in the up, down, left and right direction
3. Sketch it in!

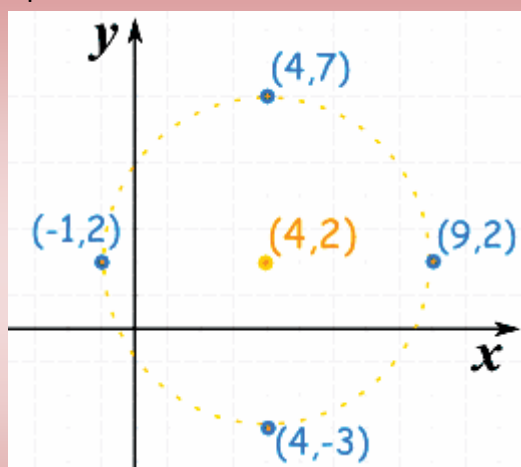
Example: Plot $(x - 4)^2 + (y - 2)^2 = 25$

The formula for a circle is $(x - a)^2 + (y - b)^2 = r^2$

So the centre is at (4, 2)

And r^2 is **25**, so the radius is $\sqrt{25} = 5$

So we can plot:



The Centre: (4, 2)

Up: $(4, 2+5) = (4, 7)$

Down: $(4, 2-5) = (4, -3)$

Left: $(4-5, 2) = (-1, 2)$

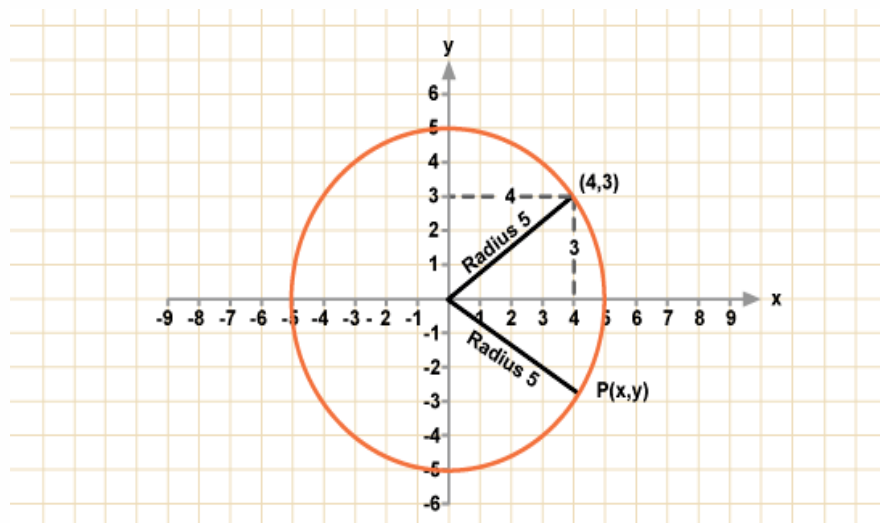
Right: $(4+5, 2) = (9, 2)$

Now, just sketch in the circle the best that you can!

For the circle below, $r = 5$, $x = 4$ and $y = 3$

So the equation of the circle is $x^2 + y^2 = 5^2$

Point (4, 3) is on the circle because $4^2 + 3^2 = 5^2$



Domain : $-5 \leq x \leq 5, x \in \mathbb{R}$

Range: $-5 \leq y \leq 5, y \in \mathbb{R}$

$P(x, y)$ is on the circle $x^2 + y^2 = 5^2$

General Form

But you may see a circle equation and **not know it!**

Because it may not be in the neat "Standard Form" above.

As an example, let us put some values to **a**, **b** and **r** and then expand it

Start with: $(x-a)^2 + (y-b)^2 = r^2$

Set (for example)
a=1, b=2, r=3: $(x-1)^2 + (y-2)^2 = 3^2$

Expand: $x^2 - 2x + 1 + y^2 - 4y + 4 = 9$

Gather like terms: $x^2 + y^2 - 2x - 4y + 1 + 4 - 9 = 0$

And we end up with this:

$$x^2 + y^2 - 2x - 4y - 4 = 0$$

It is a circle equation, but "in disguise"!

So when you see something like that think "... that **might** be a circle!"

In fact we can write it in "**General Form**" by putting constants instead of the numbers:

$$x^2 + y^2 + Ax + By + C = 0$$

Going from General Form to Standard Form

Imagine we have an equation in **General Form** (like the example above):

$$x^2 + y^2 - 2x - 4y - 4 = 0$$

How can we get it into **Standard Form** like $(x-a)^2 + (y-b)^2 = r^2$?

The answer is to Complete the Square... for **x** and for **y**:

Start with: $x^2 + y^2 - 2x - 4y - 4 = 0$

Put **xs** and **ys**
together on left: $(x^2 - 2x) + (y^2 - 4y) = 4$

Now to complete the square we take half of the middle number, square it and add it.

(Also add it to the right hand side so the equation stays in balance!)

And do it for x and y.

Do it for "x":

$$(x^2 - 2x + (-1)^2) + (y^2 - 4y) = 4 + (-1)^2$$

And for "y":

$$(x^2 - 2x + (-1)^2) + (y^2 - 4y + (-2)^2) = 4 + (-1)^2 + (-2)^2$$

Simplify:

$$(x^2 - 2x + 1) + (y^2 - 4y + 4) = 9$$

Finally:

$$(x - 1)^2 + (y - 2)^2 = 3^2$$

And we have it in Standard Form

Exponential Function

This is the Exponential Function:

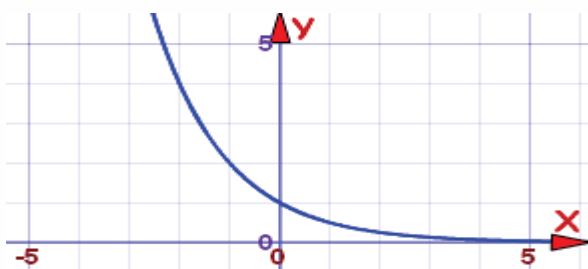
$$f(x) = a^x$$

a is any value greater than 0.

Properties depend on value of "a"

- When **a = 1**, the graph is a *horizontal line* at **y=1**
- Apart from that there are two cases to look at:

a between 0 and 1

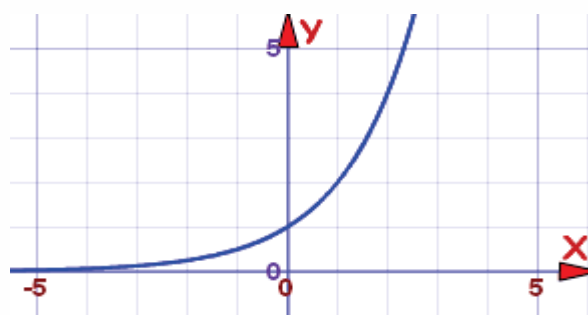


Example: $f(x) = (0.5)^x$

For a between 0 and 1

- As **x** increases, **f(x)** heads to 0
- As **x** decreases, **f(x)** heads to infinity
- Domain : $x \in \mathbb{R}$
- Range : $y > 0, y \in \mathbb{R}$
- It has a Horizontal Asymptote along the x-axis ($y=0$).

a above 1



Example: $f(x) = (2)^x$

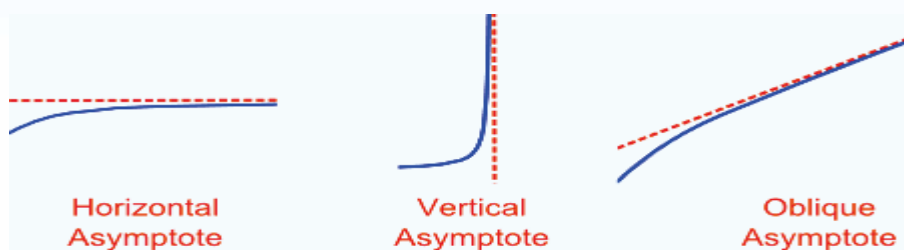
For a above 1:

- As **x** increases, **f(x)** heads to infinity
- As **x** decreases, **f(x)** heads to 0
- Domain : $x \in \mathbb{R}$
- Range : $y > 0, y \in \mathbb{R}$
- It has a Horizontal Asymptote along the x-axis ($y=0$).

Note: (Only) Range will change with the equation of the exponential function

Asymptote

A **line** that a curve approaches, as it heads towards infinity:



It can be in a negative direction, the curve can approach from any side (such as from above or below for a horizontal asymptote)

The important point is that:

The **distance** between the curve and the asymptote **tends to zero** as they head to infinity (or $-\infty$)

EXERCISE

4.2

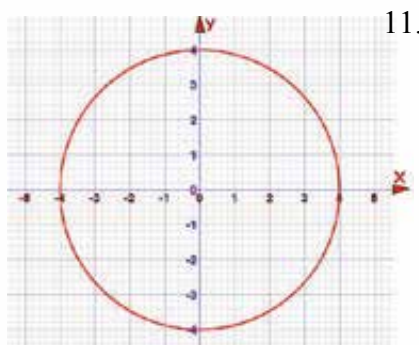
- Sketch the graph of $y = 4x^2$ and $y = 6x^2$ on the same pair of axis .
- Sketch the graph of $y = 5x^2$ and $y = -5x^2$ on the same pair of axis.
- Sketch the graph of the following quadratic equation showing the intercets.
[i] $y = x^2 - 9$ [ii] $y = x^2 - 3x$ [iii] $y = 4x - x^2$ [iv] $y = 2x - x^2$
- Sketch the grph of the following giving the coordinates of the vertex and the equation of the axis of symmetry.
[i] $y = (x - 2)^2 + 3$ [ii] $y = (x + 1)^2 + 3$ [iii] $y = 2(x - 3)^2 - 3$
[iv] $y + 1 = (x - 1)^2$ [v] $y - 4 = 3(x + 1)^2$ [vi] $y = 2 - 2(x - 4)^2$

5. What is the coordinates of the vertex of the parabola $f(x) = 2x^2 + 8x - 12$?
6. Change the following equations in the form $f(x) = a(x - h)^2 + k$, using perfect square and sketch its graph showing the intercepts and vertex.

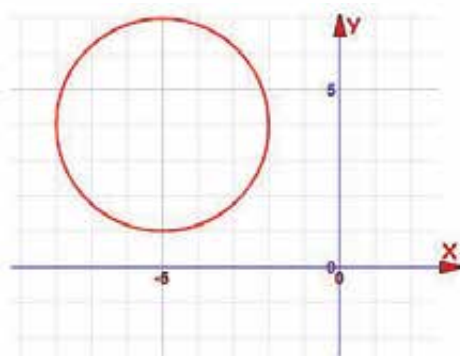
[i] $y = x^2 - 2x + 3$	[ii] $y = x^2 + 6x - 2$	[iii] $y = x^2 - 3x - 3$
[iv] $y = 2x^2 - 4x + 3$	[v] $y = 3x^2 - 2x - 1$	[vi] $y = 3x^2 - 5x + 3$
7. Sketch the graph of $y = 3x^3$ and $y = -3x^3$ on the same pair of axis
8. Sketch the graph of the following cubic functions:

[a] $y = (x-1)(x+2)(x-3)$	[b] $y = (x+3)(x+2)(x-1)$
[c] $y = -(x-3)(x+2)(x+3)$	[d] $y = y = x(x+2)(x-3)$
[e] $y = (x-1)^2(x-3)$	[f] $y = (x-1)^2(x+2)$
[g] $y = (2-x)^2(x-3)$	[h] $y = (1-x^2)(2-x)$
9. Sketch the graph of the following exponential functions

[i] $y = 3^x$	[ii] $y = 5^x$	[iii] $y = (\frac{5}{8})^x$	[iv] $y = \frac{1}{2}^x$
---------------	----------------	-----------------------------	--------------------------
10. Write down the equation of the circle in standard Form



11. Write down the equation of the circle in standard Form



- 12 . Plot the following circles

[a] $x^2 + y^2 = 16$

[c] $(x - 2)^2 + (y - 5)^2 = 81$

[e] $(x+2)^2 + (y + 4)^2 = 64$

[g] $x^2 + y^2 - 2x - 4y - 4 = 0$

[b] $2x^2 + 2y^2 = 100$

[d] $(x+1)^2 + (y - 2)^2 = 125$

[f] $(x+1)^2 + (y - 2)^2 = 125$

13 . What is the centre of the circle $(x - 5)^2 + (y + 3)^2 = 49$?

14. What is the radius of the circle $(x + 2)^2 + (y - 4)^2 = 36$?

15. What is the centre of the circle $x^2 + y^2 + 8x - 12y + 27 = 0$?

16. What is the radius of the circle $x^2 + y^2 - 6x + 14y - 23 = 0$?

17. The equation of a circle in Standard Form is $(x + 11)^2 + (y - 9)^2 = 16$.

What is the equation of the circle in General Form?

18. The equation of a circle in Standard Form is $(x - 2.5)^2 + (y + 3.5)^2 = 10$.

What is the equation of the circle in General Form?

19. The General Form of the equation of a circle is $x^2 + y^2 + x + 3y - 6.5 = 0$.

Write the equation in Standard Form.

STRAND 5

COORDINATE GEOMETRY

LEARNING OUTCOMES

Sub Strand

Writing Coordinates

- Find distance between two points
- Find the midpoint
- Determine equation of a line.

- Able to identify the distance formula and calculate the distance between two points.
- Able to use the formula and find the midpoint
- Able to determine the gradient from two given points.
- Able to find equation of a line from y intercept and gradient, two points on the line and angle with x axis and y intercept.
- To know the relationship between the gradients of parallel and perpendicular lines

The coordinates are always written in a certain order:

- the horizontal distance first,
- then the vertical distance.
- This is called an "**ordered pair**" (a **pair** of numbers in a special **order**)

And usually the numbers are separated by a comma, and parentheses are put around the whole thing like this: (3, 2)

Example: (3,2) means 3 units to the right, and 2 units up

Example: (0, 5) means 0 units to the right, and 5 units up.

In other words, only 5 units up.

The Origin

The point (0,0) is given the special name "The Origin", and is sometimes given the letter "O".

Abscissa and Ordinate

You may hear the words "Abscissa" and "Ordinate" ... they are just the x and y values: Abscissa: the horizontal ("x") value in a pair of coordinates: how far **along** the point is Ordinate: the vertical ("y") value in a pair of coordinates: how far **up or down** the point is "**Cartesian**"...

The History of Coordinate Geometry

Geometry's origins go back to approximately 3,000 BC in ancient Egypt. Ancient Egyptians used an early stage of geometry in several ways, including the surveying of land, construction of pyramids, and astronomy. Around 2,900 BC, ancient Egyptians began using their knowledge to construct pyramids with four triangular faces and a square base.

Euclid's Element

The next great advancement in geometry came from Euclid in 300 BC when he wrote a text titled 'Elements.' In this text, Euclid presented an ideal axiomatic form (now known as Euclidean geometry) in which propositions could be proven through a small set of statements that are accepted as true. In fact, Euclid was able to derive a great portion of planar geometry from just the first five postulates in 'Elements.' These postulates are listed below:

- (1) A straight line segment can be drawn joining any two points.
- (2) A straight line segment can be drawn joining any two points.
- (3) Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.
- (4) All right angles are congruent.
- (5) If two lines are drawn which intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended infinitely. Euclid's fifth postulate is also known as the parallel postulate.

René Descartes' Coordinate Geometry

The next tremendous advancement in the field of geometry occurred in the 17th century when René Descartes discovered coordinate geometry. Coordinates and equations could be used in this type of geometry in order to illustrate proofs. The creation of coordinate geometry opened the doors to the development of calculus and physics.

The Development of Non-Euclidean Geometry

In the 19th century, Carl Friedrich Gauss, Nikolai Lobachevsky, and János Bolyai formally discovered non-Euclidean geometry. In this kind of geometry, four of Euclid's first five postulates remained consistent, but the idea that parallel lines do not meet did not stay true. This idea is a driving force behind elliptical geometry and hyperbolic geometry.

Distance Formula

Used to find the distance between any two points.

1. Use $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Label the ordered pairs.
3. Substitute the values into the formula.
4. Use order of operations to simplify.

Examples:

(4, 5) & (-2, 6)

(-14, -4) & (1, 8)

(-2, 10) & (5, -3)

How to Determine if Three Given Points are the Vertices of a Right Triangle

1. Label the points using A, B, and C.
2. Find the 3 lengths (AB, AC, BC) between the points using the distance formula.
 - If the lengths cannot be square rooted evenly, leave in radical form.
3. Apply the Pythagorean Theorem ($a^2 + b^2 = c^2$)
 - Substitute the two shortest lengths for "a" and "b".
 - Substitute the longest length for "c".
 - Simplify the equation.
 - If the result is true, the three points make a right triangle.
 - If the result is false, the three points do not make a right triangle.

Midpoint Formula

Used to find the middle of a segment that connects two points.

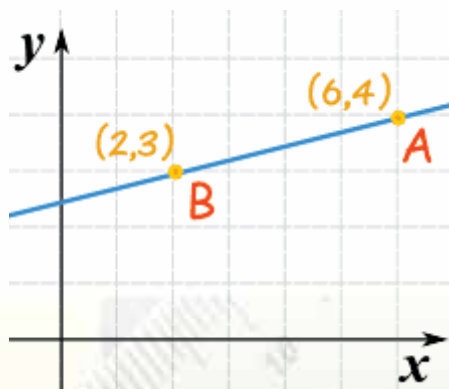
- $$1. \text{ Use } mp = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$
- Label the ordered pairs (x_1, y_1) and (x_2, y_2) .
 - Substitute the values into the formula.
 - Use order of operations to simplify.

Examples:

$(-14, -4)$ & $(1, 8)$

$(-2, 10)$ & $(5, -3)$

Finding Slope (or Gradient) from 2 Points



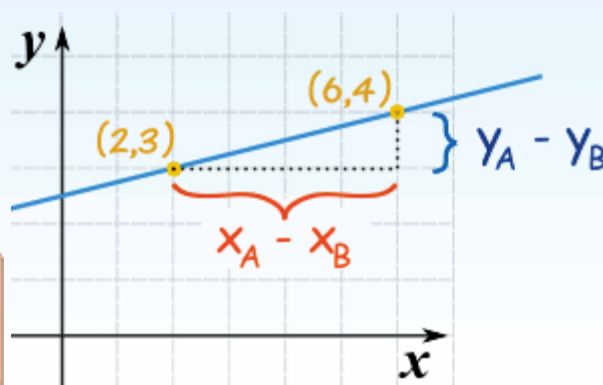
What is the slope (or gradient) of this line?

We know two points:

- point "A" is $(6, 4)$ (x is 6 when y is 4)
- point "B" is $(2, 3)$ (x is 2 when y is 3).

The formula is:

$$\text{Slope } m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_A - y_B}{x_A - x_B}$$



So what we do is

- subtract the Y values,
- subtract the X values
- then divide

Like this:

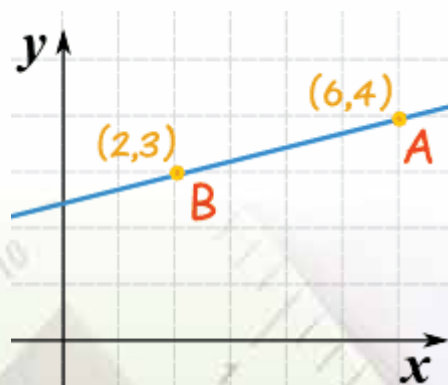
$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{4 - 3}{6 - 2} = \frac{1}{4} = 0.25$$

It doesn't matter which point comes first, it still works out the same. Try swapping the points:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{3 - 4}{2 - 6} = \frac{-1}{-4} = 0.25$$

Finding an Equation from 2 Points

Now you know how to find the **slope**, let us look at finding a whole **equation**.



What is the equation of this line?

The easiest method is to start with the "point-slope" formula:

$$y - y_1 = m(x - x_1)$$

We can choose **any point** on the line as being point "1", so let us just use point (2,3):

$$y - 3 = m(x - 2)$$

Use the formula from above for the slope "m":

$$\text{Slope } m = \frac{\text{change in } y}{\text{change in } x} = \frac{4 - 3}{6 - 2} = \frac{1}{4}$$

And we have:

$$y - 3 = (1/4)(x - 2)$$

That is an acceptable answer, but we could simplify it further:

$$y - 3 = x/4 - 2/4$$

$$y = x/4 - 1/2 + 3$$

$$y = x/4 + 5/2$$

$$4y = x + 10$$

EXERCISE 5.1

1. Find the midpoint of the segment connecting the points (6,4) and (3,-4).
2. Find the midpoint of the segment connecting the points (a,b) and (3a,c).
3. M is the midpoint of $\overline{AB}$. The coordinates of A are (-2,3) and the coordinates of M are (1,0). Find the coordinates of B.
4. M is the midpoint of $\overline{AB}$. The coordinates of A are (2,3) and the coordinates of M are (4.5, 6). Find the coordinates of B.
5. $\overline{CD}$ is the diameter of a circle whose center is the point (2,1). If the coordinates of C are (0,-2), find the coordinates of D.
6. Find the distance between the points (-4,-5) and (1,-2).
7. Find the length of the line segment whose endpoints are (-3, 4) and (5,4).
8. The point (5,4) lies on a circle. What is the length of the radius of this circle if the center is located at (3,2)?
9. Find the equation of the line passing through the points (2,4) and (-3,-6)?

Sub Strand

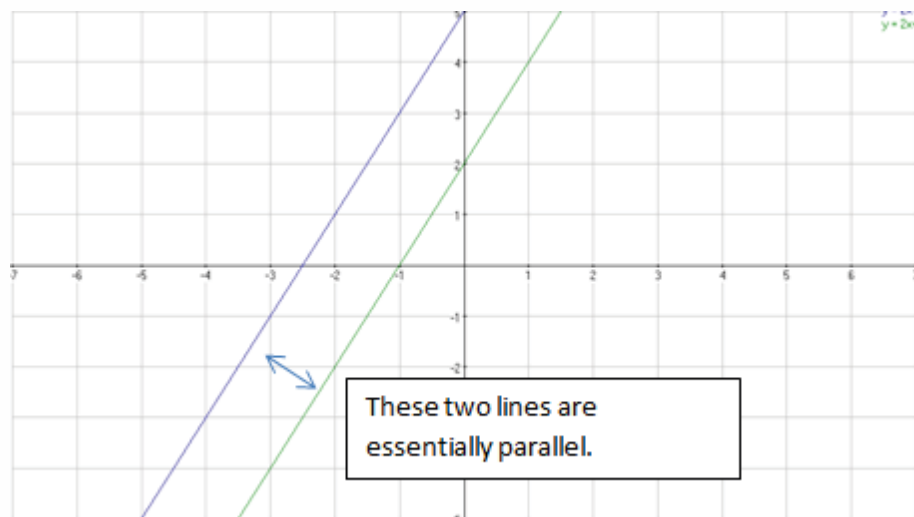
Parallel and Perpendicular Lines

- Describe the relationship of gradient of parallel and perpendicular lines
- Determine equation of parallel and perpendicular lines.

Parallel Lines

“Parallel lines have the same gradient”

Here is an example of a pair of parallel lines



These two lines are clearly parallel to each other.

Remember the form $y = ax + b$, where “a” is the gradient?

The **gradient** here is the **same**; it’s just the value of “b”, the **y intercept** that is **different**.

The rule to learn here is that for two functions $y = ax + b$ and $y_1 = a_1x_1 + b_1$,

$$a = a_1$$

The actual functions here are : $2x+2$ and $2x+5$.

As you can see, the two values of “a” are the same but the values of “b” are different.

Perpendicular Lines

If you want to calculate the perpendicular of the gradient, simply remember this:

For the linear function $y = ax + b$, $a \cdot a_1$ (where a_1 is the gradient of the perpendicular) = -1

So:

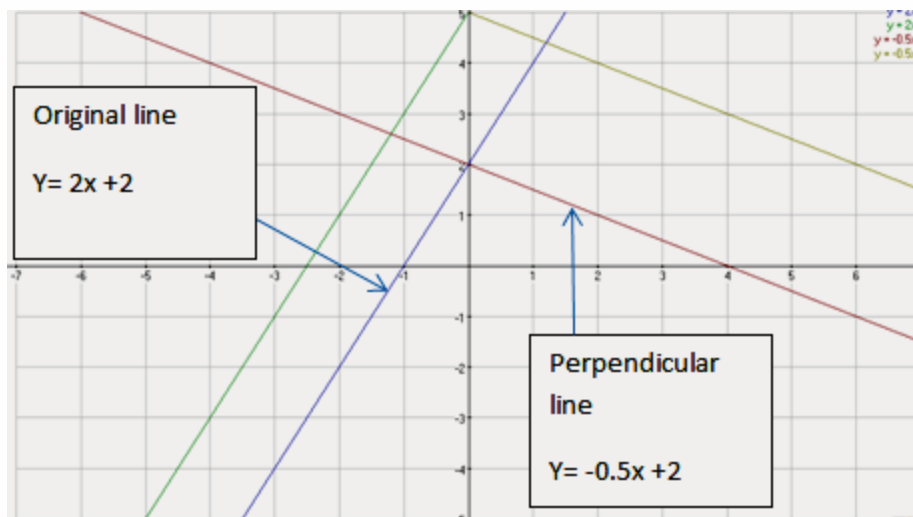
$$a \cdot a_1 = -1$$

We can rearrange this to get:

$$a_1 = \frac{-1}{a}$$

So for the function $y = 2x + 2$, the gradient of the perpendicular line is:

$$a_1 = -1/2$$



If we plot this graphically, we notice that the perpendicular line forms a right angle with the original line, as seen below.

EXERCISE

5.2

- What is the slope of a line perpendicular to $2y = -6x - 10$?
- Given $3y - 4x = 5$ and $4y + 6 = 3x$.
Are these lines parallel, perpendicular or neither? Why
- What is the equation of a line that passes through the point $(4, -5)$ and is parallel to $3x + 2y = 12$?
- What is the equation of a line that passes through the point $(-1, -2)$ and is perpendicular to $-5x = 6y + 18$?
- A line passes through the points shown in this table.
What is the slope of a line perpendicular to this line?
- Given $4y - 2x = 10$ and $-6y - 6 = -3x$. Are these lines parallel, perpendicular or neither?
- What is the equation of a line that is parallel to $y = -4$ and passes through the point $(3, 7)$?
- Write the equation of a line that is parallel to $4x + 2y = -8$ and has the same y -intercept as $-3y = -2x - 9$?
- A line passes through a point $(2, 5)$ and has a slope of -3 . What is the equation of a line perpendicular to this line through $(2, 5)$?

x	1	3	5	7
y	2	5	8	11

STRAND 6

TRIGONOMETRY

LEARNING OUTCOMES

Sub Strand

Trigonometric Ratios

- Deduce right angled triangle from a 3 dimensional object
- Use Pythagoras theorem and SOH/CAH/TOA to solve right triangles

Able to find missing sides using Pythagoras Theorem

- Able to identify sine, cosine and tangents relations to find missing sides and angles in a right angle triangle.
- Able to reduce to right angle triangle and solve
- Draw graphs of Sine, Cosine and tangents in the domain 0 to 360°

Brief History of Trigonometry

Trigonometry deals with the study of angles, triangles, and trigonometric functions. Taken from the Greek words *trigonon* (triangle) and *metria* (measure), the word literally means *triangle measurement* and the term came into use in the 17th century—the period when trigonometry, as an analytic science, started; but its real origins lie in the ancient Egyptian pyramids and Babylonian astronomy that date back to about 3000 BCE.

It is the Greek astronomer and mathematician Hipparchus of Nicaea in Bithynia (190 BCE - 120 BCE) that is often considered as the founder of the science of trigonometry. According to the Greek scholar Theon of Alexandria (335 CE - 405 CE), Hipparchus compiled a "table of chords" in a circle (a trigonometric table) in 12 books.

Regarding the six trigonometric functions: Aryabhata (476 CE - 550 CE) discovered the sine and cosine; Muhammad ibn Musa al-Khwarizimi (780 CE - 850 CE) discovered the tangent; Abu al-Wafa' Buzjani (940 CE - 988 CE) discovered the secant, cotangent, and cosecant.

Albert Girard (1595-1632), a French mathematician, was the first to use the abbreviations sin, cos, and tan in a treatise.

The history of trigonometry came about mainly due to the purposes of time keeping and astronomy.

History of Pythagoras Theorem

Over 2,500 years ago, a Greek mathematician named Pythagoras popularized the concept that a relationship exists between the hypotenuse and the legs of right triangles and that this relationship is true for *all* right triangles. The Egyptians knew of this concept, as it related to 3, 4, 5 right triangles, long before the time of Pythagoras. It was Pythagoras, however, who generalized the concept and who is attributed with its first geometrical demonstration. Thus, it has become known as the Pythagorean Theorem

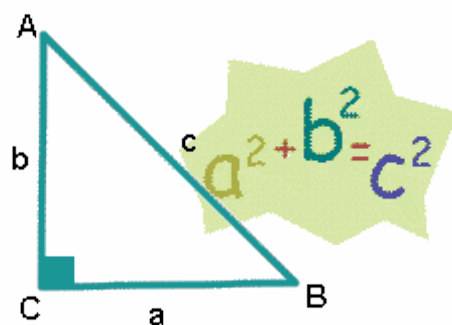
"In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs."

This relationship can be stated as:

$$c^2 = a^2 + b^2$$

for any right triangle

and is known as the
Pythagorean Theorem.



a, b are **legs**.

c is the **hypotenuse**

(c is across from the right angle).



There are certain sets of numbers that have a very special property in relation to the Pythagorean Theorem. Not only do these numbers satisfy the Pythagorean Theorem, but **any** multiples of these numbers also satisfy the Pythagorean Theorem.

Example:

The numbers **3**, **4**, and **5** satisfy the Pythagorean Theorem. If you multiply all three numbers by 2 (**6**, **8**, and **10**), these new numbers ALSO satisfy the Pythagorean theorem.

The special sets of numbers that possess this property are called Pythagorean Triples or triads.

The most common Pythagorean Triples are:

3, 4, 5

5, 12, 13

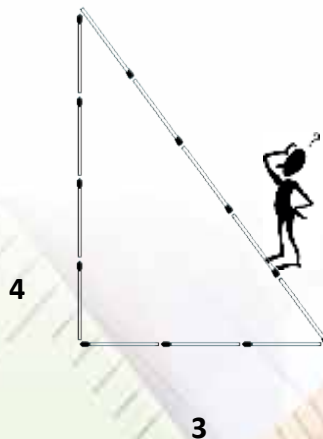
8, 15, 17



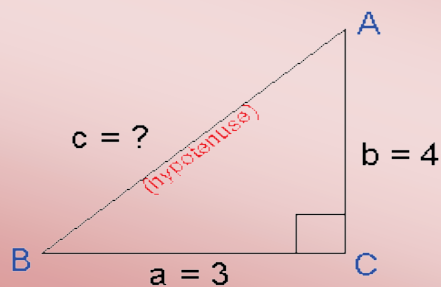
REMEMBER: The Pythagorean Theorem
ONLY works on **Right Triangles!**

Activity: Let's form a triangle

Get 12 matchsticks and form an "L" using matchsticks as the **base** and four matchsticks as the **height**. Turn the "L" into a triangle by connecting the height and base using the remaining matchsticks. How many matchsticks are needed to complete the triangle? Is there any relationship between the sides in a right angle triangle?

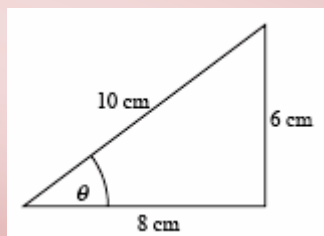


Example 1: Let's look at our matchstick triangle and confirm our results



$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ \sqrt{25} &= c \\ c &= 5 \end{aligned}$$

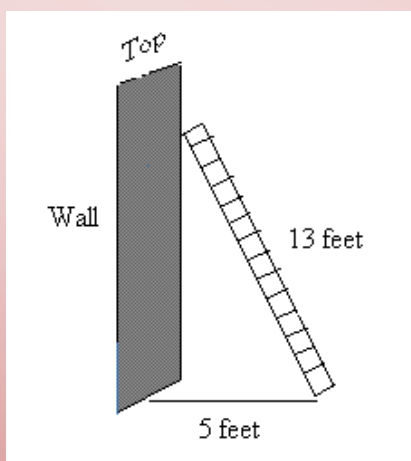
Example 2: A triangle has lengths 6, 8 and 10. Is this a right angle triangle?



$$\begin{aligned} c^2 &= a^2 + b^2 \\ 10^2 &= 6^2 + 8^2 \\ 100 &= 36 + 64 \\ 100 &= 100 \end{aligned}$$

It is a right angle triangle.

Example 3: A 13 feet ladder leans against a wall. The base of the ladder is 5 feet from the wall. Determine how high the ladder will reach up the wall?



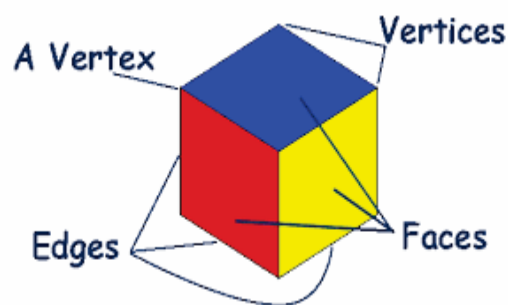
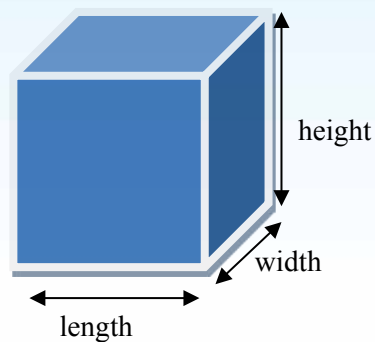
$$\begin{aligned} c^2 &= a^2 + b^2 \\ 13^2 &= 5^2 + x^2 \\ 169 &= 25 + x^2 \\ 169 - 25 &= x^2 \\ 144 &= x^2 \\ \sqrt{144} &= x \\ 12 &= x \end{aligned}$$

The ladder will reach 12 feet up the wall

The 3-4-5 triangle is a right triangle and any multiple of 3-4-5 is also a right angle triangle. For example 3-4-5 multiplied by 2 equals 6-8-10, which also satisfies the Pythagoras theorem. These are called Pythagoras Triples or Triads.

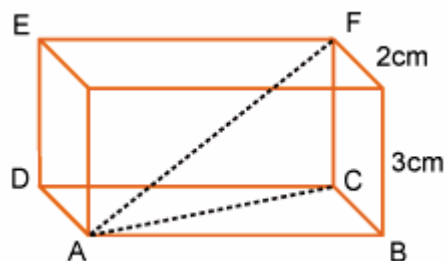
Three Dimensional Objects

3D objects have three dimensions: width, depth and height.



Example

This cuboid has side lengths of 2cm, 3cm and 6cm.



Work out the length of the diagonal AF.

Solution

First use Pythagoras' theorem in triangle ABC to find length AC.

$$AC^2 = 6^2 + 3^2$$

$$AC = \sqrt{45}$$

You do not need to find the root as we will need to square it in the following step. Next we use Pythagoras' theorem in triangle ACF to find length AF.

$$AF^2 = AC^2 + CF^2$$

$$AF^2 = 45 + 2^2$$

$$AF = \sqrt{49}$$

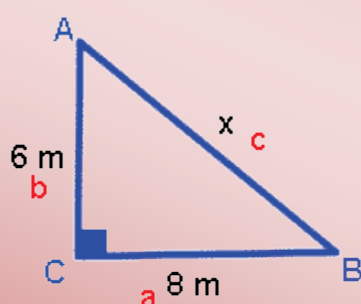
$$AF = 7\text{cm}$$

Real life Applications of Pythagoras Theorem:

1. **Travelling:** The shortest distance is always a direct path
2. **Architecture & Buildings:** Constructing Triangular Roofs, Painting Walls
3. **Earthquakes:** Geologists use Pythagoras theorem to locate earthquakes.

Can you think of some more?

Example 1:



Find x .

Answer: 10 m

$$c^2 = a^2 + b^2$$

$$x^2 = 64 + 36$$

$$x^2 = 100$$

$$\sqrt{x^2} = \sqrt{100}$$

$$x = 10$$

This problem could also be solved using the Pythagorean Triple 3, 4, 5. Since 6 is 2 times 3, and 8 is 2 times 4, then x must be 2 times 5.

Example 2:

A triangle has sides 6, 7 and 10.
Is it a right triangle?

Let $a = 6$, $b = 7$ and $c = 10$. The longest side MUST be the hypotenuse, so $c = 10$. Now, check to see if the Pythagorean Theorem is true.

$$c^2 = a^2 + b^2$$

$$10^2 ? 6^2 + 7^2$$

$$100 ? 36 + 49$$

$$100 \neq 85$$

Since the Pythagorean Theorem is NOT true, this triangle is **NOT** a right triangle.

Trigonometric Ratios

Sine, Cosine and Tangent

Sine, **Cosine** and **Tangent** are the three main functions in trigonometry.

They are often shortened to **sin**, **cos** and **tan**.

The calculation is simply **one side of a right angled triangle divided by another side** ... you just have to know which sides, and that is where "sohcahtoa" helps.

For a triangle with an angle θ , the functions are calculated this way:

Sine Function: **..soh...** $\sin(\theta) = \text{opposite} / \text{hypotenuse}$

Cosine Function: **...cah...** $\cos(\theta) = \text{adjacent} / \text{hypotenuse}$

Tangent Function: **...toa...** $\tan(\theta) = \text{opposite} / \text{adjacent}$

SOH/CAH/TOA

Sohca...*what?* Just an **easy way to remember** how Sine, Cosine and Tangent work:

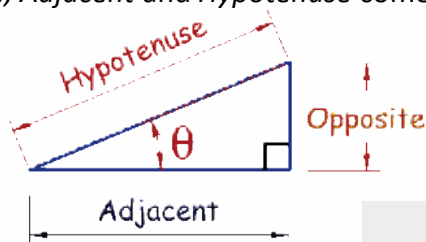
Soh... Sine = Opposite / Hypotenuse

Cah... Cosine = Adjacent / Hypotenuse

Toa... Tangent = Opposite / Adjacent

Right Triangle

The names *Opposite*, *Adjacent* and *Hypotenuse* come from the right triangle:



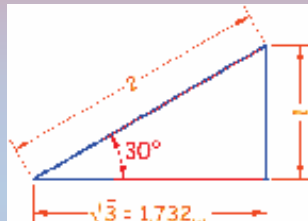
- "Opposite" is opposite to the angle θ .
- "Adjacent" is adjacent (next to) to the angle θ
- "Hypotenuse" is the longest side.



Adjacent is always next to the angle
(and **opposite** is on the other side of the angle)

Example: What are the sine, cosine and tangent of 30° ?

The classic 30° triangle has a hypotenuse of length **2**, an opposite side of length **1** and an adjacent side of **$\sqrt{3}$** , like this:



Now we know the lengths, we can calculate the functions:

Sine	... soh...	$\sin(30^\circ) = 1 / 2 = 0.5$
Cosine	... cah...	$\cos(30^\circ) = 1.732 / 2 = 0.866$
Tangent	... toa...	$\tan(30^\circ) = 1 / 1.732 = 0.577$

(get your calculator out and check them!)

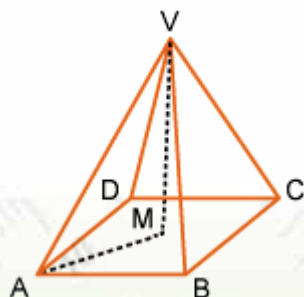
EXERCISE 6.1

- Find the length of the diagonal of a rectangle with a length of 10 centimetres (cm) and a width of 7 cm.
- Find the width of a rectangle whose diagonal is 12 cm and whose length is 10 cm.
- How long must a guywire be to run from the top of a 20 m pole to a point on the ground 8 m from the base of the pole?
- The base of a 15m ladder is 5 m away from a wall. How high from the floor is the top of the ladder?
- The length of one leg of a right triangle is 3 in. more than the other. If the length of the hypotenuse is 15 in., what are the lengths of the two legs?

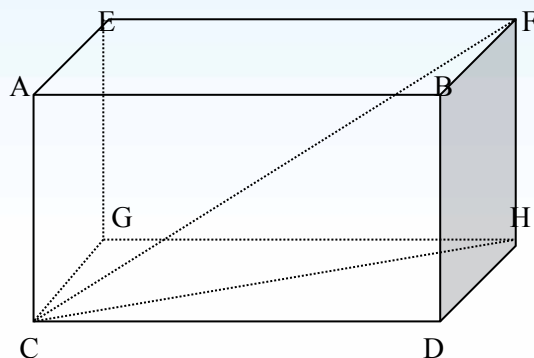
6. Two friends are meeting at a playground. One friend is located on the south-west corner and the other is located on the north-east corner of the playground. There are two paths by which they can meet. With the help of Pythagoras' theorem calculate the shortest distance between the meeting points of the two friends.

Solve the following word problems. For each question, draw a diagram to help you.

1. An airplane is flying at an altitude of 6000 m over the ocean directly toward a coastline. At a certain time, the angle of depression to the coastline from the airplane is 14° . How much farther (to the nearest kilometre) does the airplane have to fly before it is directly above the coastline?
2. From a horizontal distance of 80.0 m, the angle of elevation to the top of a flagpole is 18° . Calculate the height of the flagpole to the nearest tenth of a metre.
3. A 9.0 m ladder rests against the side of a wall. The bottom of the ladder is 1.5 m from the base of the wall. Determine the measure of the angle between the ladder and the ground, to the nearest degree.
4. The angle of elevation of the sun is 68° when a tree casts a shadow 14.3 m long. How tall is the tree, to the nearest tenth of a metre?
5. A wheelchair ramp is 4.2 m long. It rises 0.7 m. What is its angle of inclination to the nearest degree?
6. A person flying a kite has released 176 m of string. The string makes an angle of 27° with the ground. How high is the kite? How far away is the kite horizontally? Give your answer to the nearest metre.
7. Work out the vertical height (VM) in this square-based pyramid. The square base ABCD has side lengths of 5cm. Lengths VA, VB, VC and VD are each 8cm. M is the mid-point of the square base.



8. A room is in the shape of a cuboid. It's floor measures 7.2m by 9.6m and its height is 3.5m.
- Calculate the length of CH.
 - Calculate the length of CF.
 - Calculate the angle that FCH



Angles and Radian Measure

This section will cover how angles are drawn and also arc length and rotations.

Angles are measured in a couple of different ways. The first unit of measurement is a degree in which (degrees) is equal to one revolution.

Another unit of measurement for angles is **radians**.

In radians, 2π is equal to one revolution. So a conversion between radians and degrees is

$$360^\circ = 2\pi \text{ radians or } 180^\circ = \pi \text{ radians}$$

When converting from **degrees to radians**:

Multiply your degrees by $\frac{\pi}{180^\circ}$.

When converting from **radians to degrees**:

Multiply your radians by $\frac{180^\circ}{\pi}$.

Convert 60° to radians.

We will take 60° and multiply it by $\frac{\pi}{180^\circ}$ and you will get: $60 \cdot \frac{\pi}{180^\circ}$

This reduces to $\frac{\pi}{3}$

Convert 270° to radians.

Since 180° equates to π , then:

$$\frac{270}{1} \times \frac{\pi}{180} = \frac{\cancel{270}^3}{1} \times \frac{\pi}{\cancel{180}_2} = \frac{3\pi}{2}$$

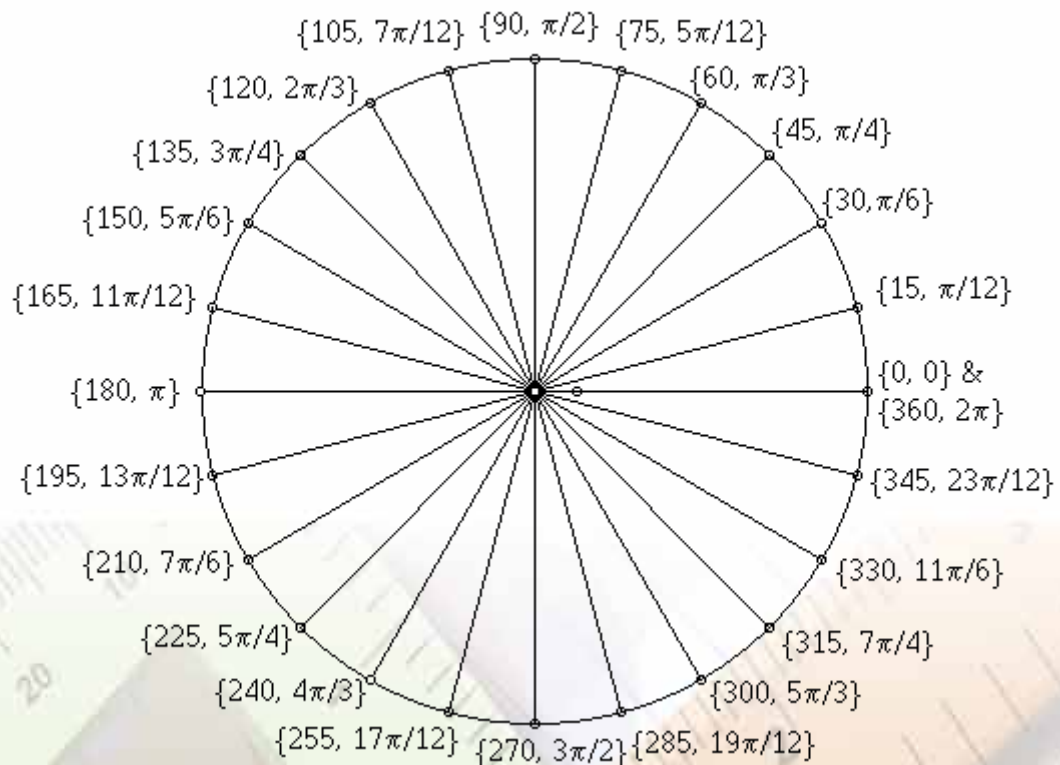
The equivalent angle is $\frac{3\pi}{2}$

Convert $\frac{\pi}{6}$ radians to degrees.

$$\frac{\pi}{6} \times \frac{180}{\pi} = \frac{\cancel{\pi}}{6} \times \frac{180^{\cancel{30}}}{\cancel{\pi}} = 30$$

The equivalent angle is 30°

The figure below gives the relationship between degrees and radians for the most common angles in the unit circle measured in the counter clockwise direction from the point to the right of the vertex. The form of the ordered pair is {degree measure, radian measure}



SINE, COSINE AND TANGENT GRAPH

The graphs in this section are probably the most commonly used in all areas of science and engineering. They are used for modelling many different natural and mechanical phenomena (populations, waves, engines, acoustics, electronics, UV intensity, growth of plants and animals, etc).

The trigonometric graphs in this chapter are periodic, which means the shape repeats itself exactly after a certain amount of time. Anything that has a regular cycle (like the tides, temperatures, rotation of the earth, etc) can be modelled using a sine or cosine curve.

1. Graphs of $y = a \sin x$ and $y = a \cos x$, talks about amplitude. Amplitude is a indication of how much energy a wave contains.
2. Graphs of $y = a \sin bx$ and $y = a \cos bx$ introduces the period of a trigonometric graph.
3. Graphs of $y = a \sin(bx + c)$ and $y = a \cos(bx + c)$ helps you to understand the displacement (or phase shift) of a trigonometric curve

Graphs of $y = a \sin x$ and $y = a \cos x$

• The Sine Curve

$y = a \sin t$

The sine curve occurs naturally when we are examining waves. When waves have more energy, they go up and down more vigorously. We say they have greater **amplitude**.

Let's investigate the shape of the curve $y = a \sin t$ and see what the concept of "**amplitude**" means.

Amplitude

The " a " in the expression $y = a \sin x$ represents the **amplitude** of the graph. It is an indication of how much **energy** the wave contains.

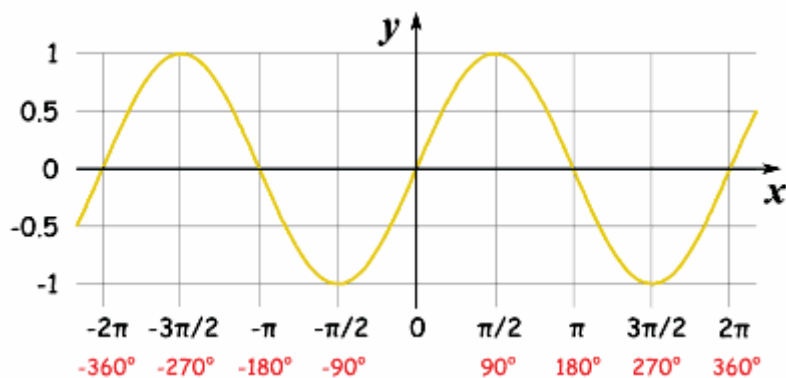
The amplitude is the distance from the "resting" position (otherwise known as the **mean value** or **average value**) of the curve.

Amplitude is always a **positive** quantity. We could write this using absolute value signs. For the curve $y = a \sin x$, amplitude = $|a|$

Sine Graph

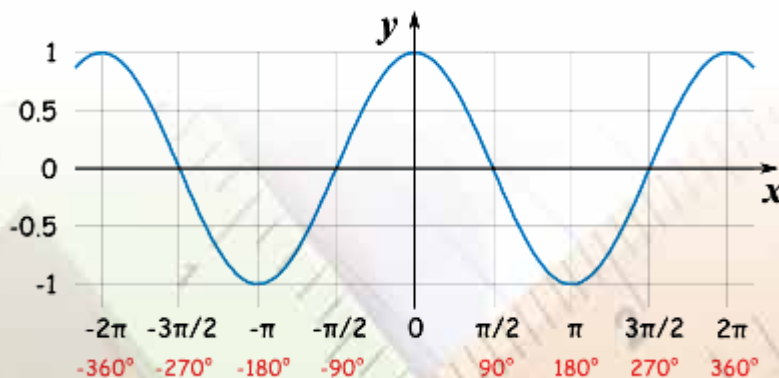
The Sine Function has this beautiful up-down curve (which repeats every 2π radians, or 360°).

It starts at 0, heads up to 1 by $\pi/2$ radians (90°) and then heads down to -1.



- ## Graph of Cosine

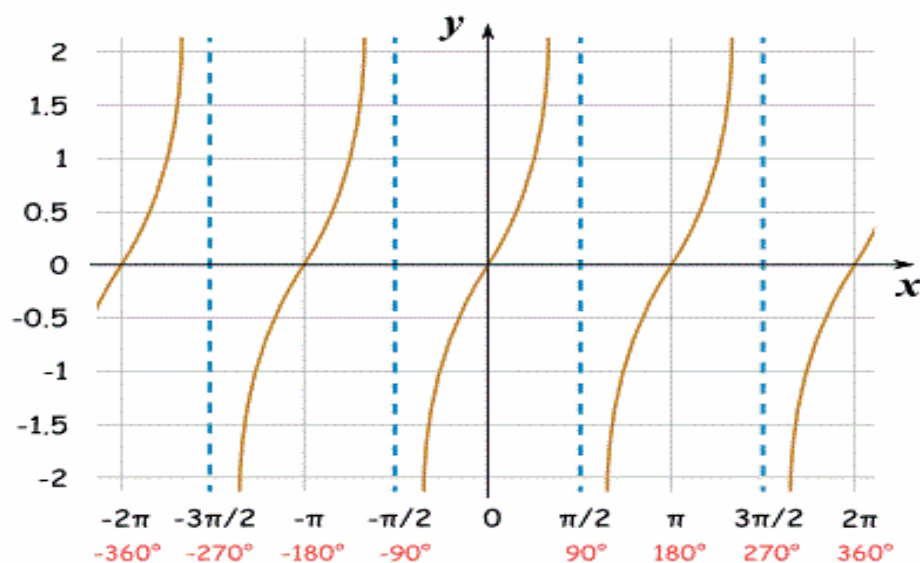
Cosine is just like Sine, but it starts at 1 and heads down until π radians (180°) and then heads up again.



• Graph of Tangent Function

The Tangent function has a completely different shape ... it goes between negative and positive Infinity, crossing through 0 (every π radians, or 180°), as shown on this plot.

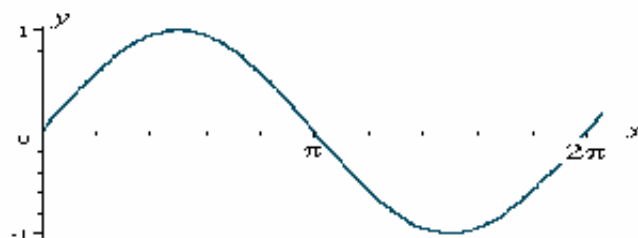
At $\pi/2$ radians, or 90° (and $-\pi/2$, $3\pi/2$, etc) the function is officially undefined, because it could be positive Infinity or negative Infinity.



Graph of Sine x - with varying amplitudes

We start with $y = \sin x$.

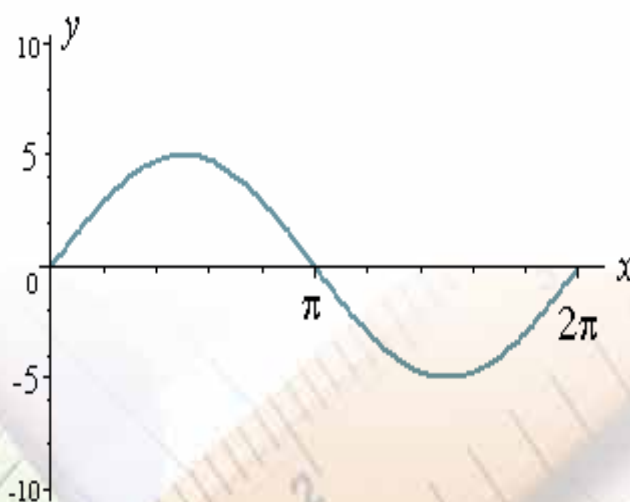
It has **amplitude** = 1 and **period** = 2π .



Now let's look at the graph of $y = 5 \sin x$.

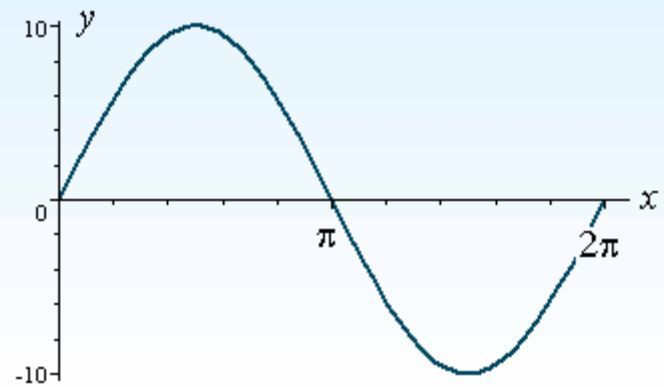
This time we have amplitude = 5 and period = 2π .

(I have used a different scale on the y-axis.)



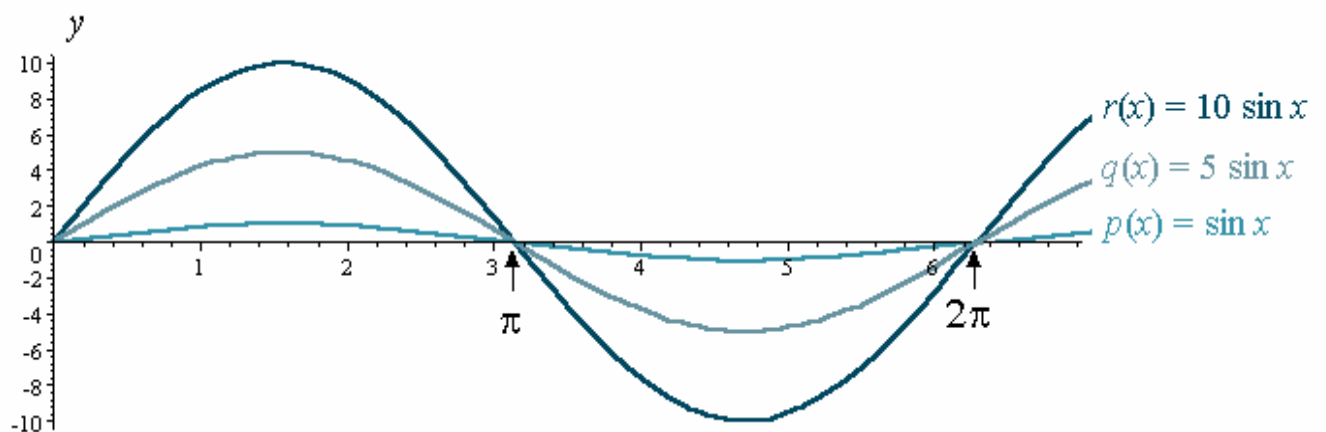
And now for $y = 10 \sin x$.

Amplitude = 10 and period = 2π .



For comparison, and using the same y -axis scale, here are the graphs of $p(x) = \sin x$, $q(x) = 5 \sin x$ and $r(x) = 10 \sin x$ on the one set of axes.

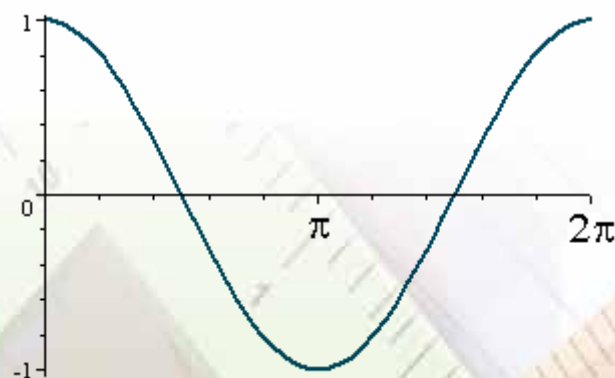
Note that the graphs have the same **period** (which is 2π) but different **amplitude**.



Graph of Cosine x - with varying amplitudes

Now let's see what the graph of $y = a \cos x$ looks like. This time the angle is measured from the positive vertical axis.

Now let's have a look at the graph of the simplest cosine curve, $y = \cos x (= 1 \cos x)$

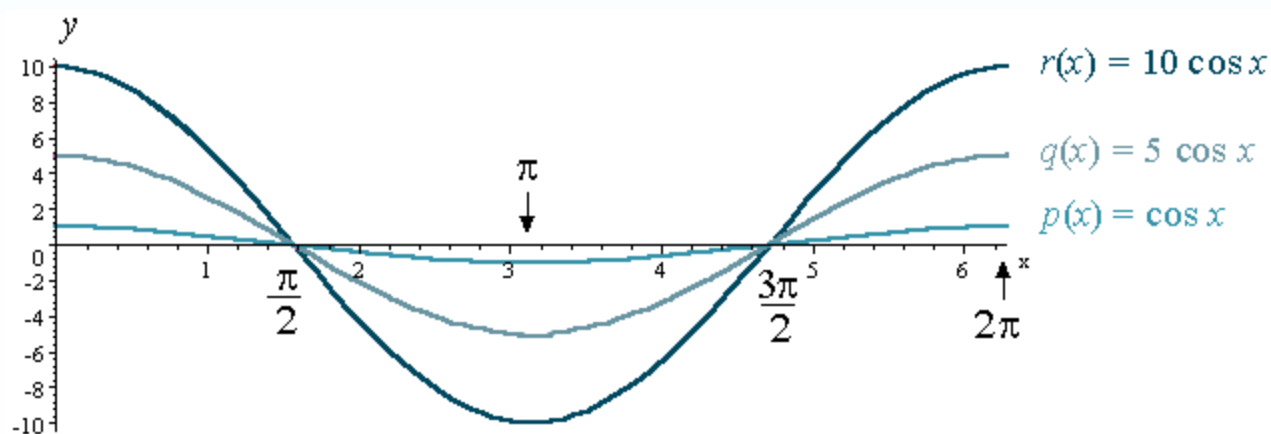


We note that the **amplitude** = 1 and **period** = 2π .

Similar to what we did with $y = \sin x$ above, we now see the graphs of

- $p(x) = \cos x$
- $q(x) = 5 \cos x$
- $r(x) = 10 \cos x$

on one set of axes, for comparison:



Note: For the cosine curve, just like the sine curve, the **period** of each graph is the same (2π), but the **amplitude** has changed

Graphs of $y = a \sin bx$ and $y = a \cos bx$

The variable b in both of the graph types

- $y = a \sin bx$
- $y = a \cos bx$

affects the **period** (or wavelength) of the graph. The **period** is the distance (or time) that it takes for the sine or cosine curve to begin repeating again.

Formula for Period

The relationship between b and the **period** is given by:

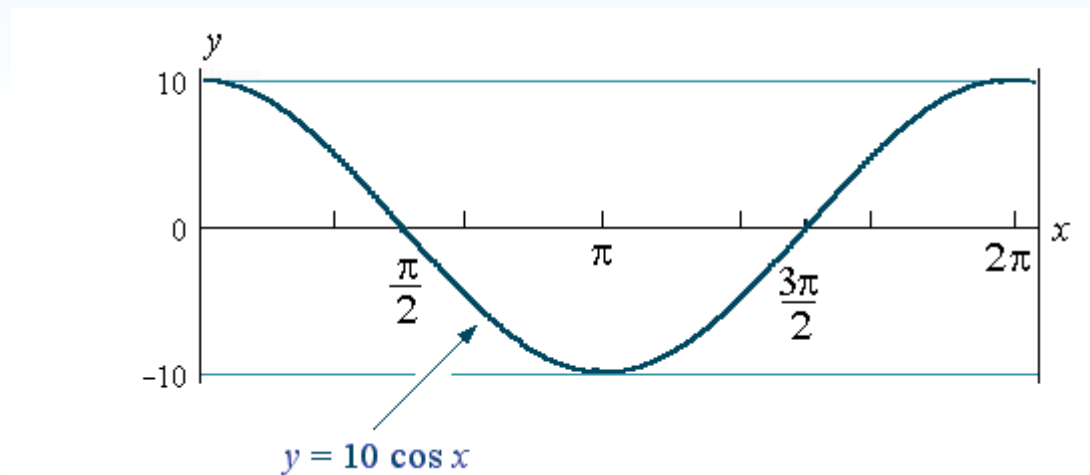
$$\text{Period} = \frac{2\pi}{b}$$

Note: As b gets larger, the period decreases

Changing the Period

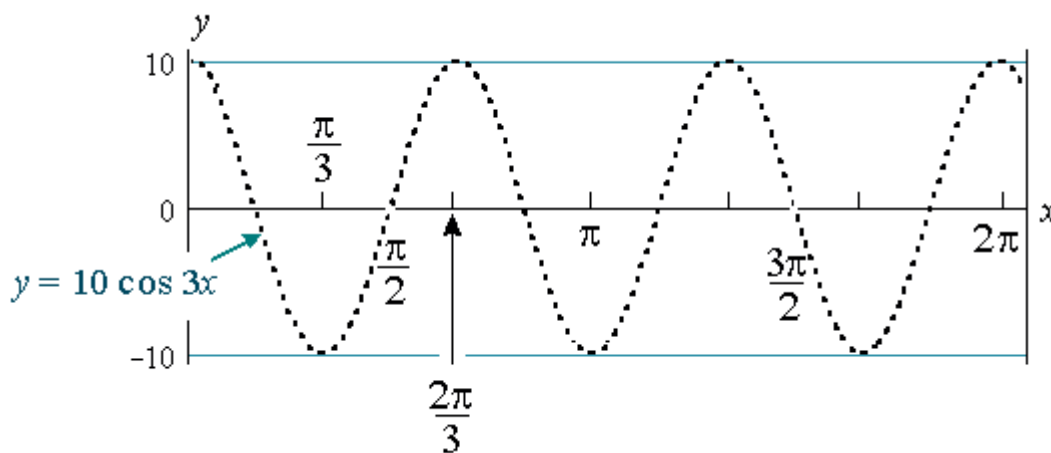
Now let's look at some still graphs to see what's going on.

The graph of $y = 10 \cos x$, which we learned about in the last section, sine and cosine curves, is as follows.



As we learned, the **period** is 2π .

Next we see $y = 10 \cos 3x$. Note the 3 inside the cosine term.

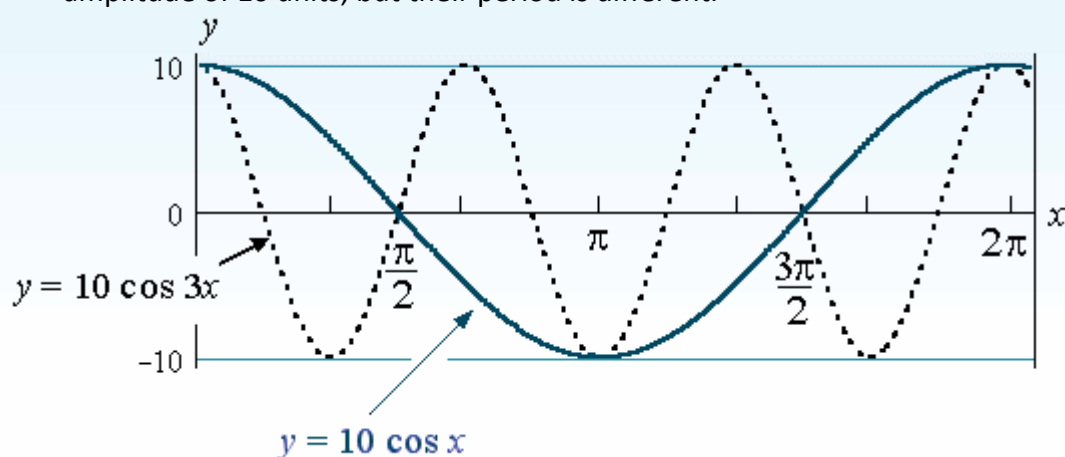


Notice that the period is different. (However, the amplitude is 10 in each example.)

This time the curve starts to repeat itself at $x = \frac{2\pi}{3}$. This is consistent with the formula we met above:

$$\text{Period} = 2\pi/b$$

Now let's view the 2 curves on the same set of axes. Note that both graphs have an amplitude of 10 units, but their period is different.



The number b tells us the number of cycles in each 2π .

For $y = 10 \cos x$, there is **one** cycle between 0 and 2π (because $b = 1$).

For $y = 10 \cos 3x$, there are **3** cycles between 0 and 2π (because $b = 3$).

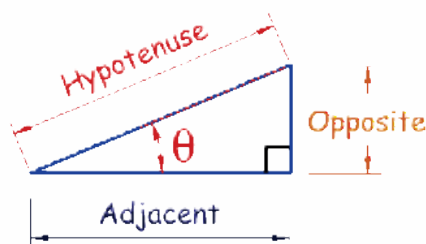
Sine, Cosine and Tangent in Four Quadrants

The three main functions in trigonometry are Sine, Cosine and Tangent

They are easy to calculate:

Divide the length of one side of a right angled triangle by another side

But we must know which sides!



For an angle θ , the functions are calculated this way:

Sine Function: $\sin(\theta) = \text{Opposite} / \text{Hypotenuse}$

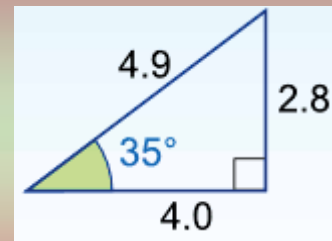
Cosine Function: $\cos(\theta) = \text{Adjacent} / \text{Hypotenuse}$

Tangent Function: $\tan(\theta) = \text{Opposite} / \text{Adjacent}$

Example: What is the sine of 35°?

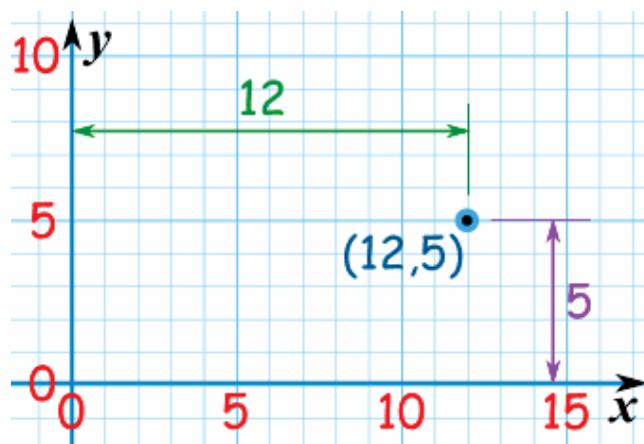
Using this triangle (lengths are only to one decimal place):

$$\sin(35^\circ) = \text{Opposite} / \text{Hypotenuse} = 2.8/4.9 = 0.57...$$



Cartesian Coordinates

Using Cartesian Coordinates we mark a point on a graph by **how far along** and **how far up** it is:



The point **(12, 5)** is 12 units along and 5 units up.

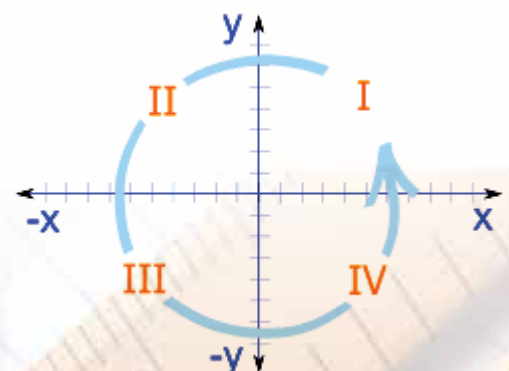
Four Quadrants

When we include **negative values**, the x and y axes divide the space up into 4 pieces:

Quadrants I, II, III and IV

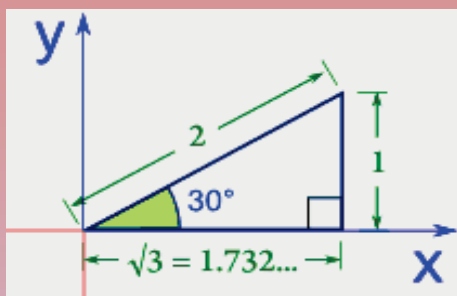
(They are numbered in a counter-clockwise direction)

- In **Quadrant I** both x and y are positive,
- in **Quadrant II** **x is negative** (y is still positive),
- in **Quadrant III** **both x and y are negative**, and
- in **Quadrant IV** x is positive again, and **y is negative**.



Now let us look at what happens when we place a 30° triangle in each of the 4 Quadrants. In Quadrant I everything is normal, and Sine, Cosine and Tangent are all positive:

Example: The sine, cosine and tangent of 30°

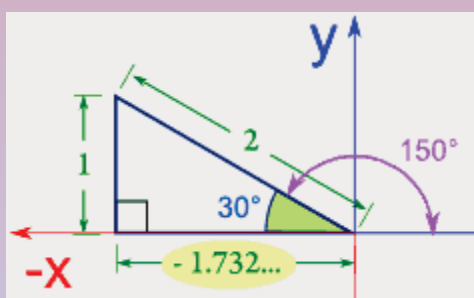


Sine: $\sin(30^\circ) = 1 / 2 = 0.5$

Cosine: $\cos(30^\circ) = 1.732 / 2 = 0.866$

Tangent: $\tan(30^\circ) = 1 / 1.732 = 0.577$

But in Quadrant II, the **x direction is negative**, and both cosine and tangent become negative:

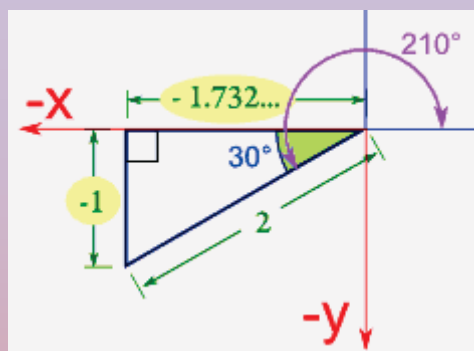


Sine: $\sin(150^\circ) = 1 / 2 = 0.5$

Cosine: $\cos(150^\circ) = -1.732 / 2 = -0.866$

Tangent: $\tan(150^\circ) = 1 / -1.732 = -0.577$

In Quadrant III, sine and cosine are negative:



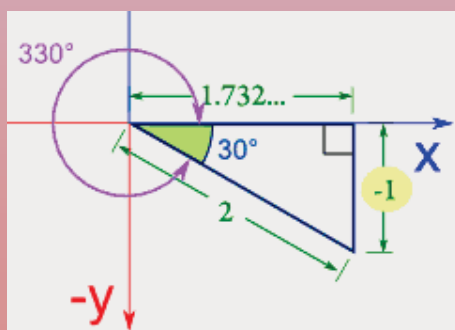
Sine: $\sin(210^\circ) = -1 / 2 = -0.5$

Cosine: $\cos(210^\circ) = -1.732 / 2 = -0.866$

Tangent: $\tan(210^\circ) = -1 / -1.732 = 0.577$

Note: Tangent is **positive** because dividing a negative by a negative gives a positive.

In Quadrant IV, sine and tangent are negative:



Sine: $\sin(330^\circ) = -1 / 2 = -0.5$

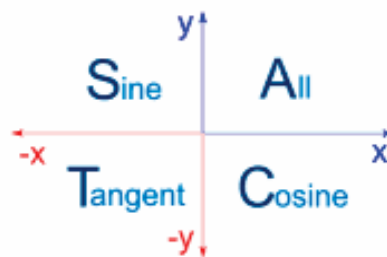
Cosine: $\cos(330^\circ) = 1.732 / 2 = 0.866$

Tangent: $\tan(330^\circ) = -1 / 1.732 = -0.577$

There is a pattern! Look at when Sine Cosine and Tangent are **positive** ...

- **All** three of them are positive in **Quadrant I**
- **Sine** only is positive in **Quadrant II**
- **Tangent** only is positive in **Quadrant III**
- **Cosine** only is positive in **Quadrant IV**

This can be shown even easier by:

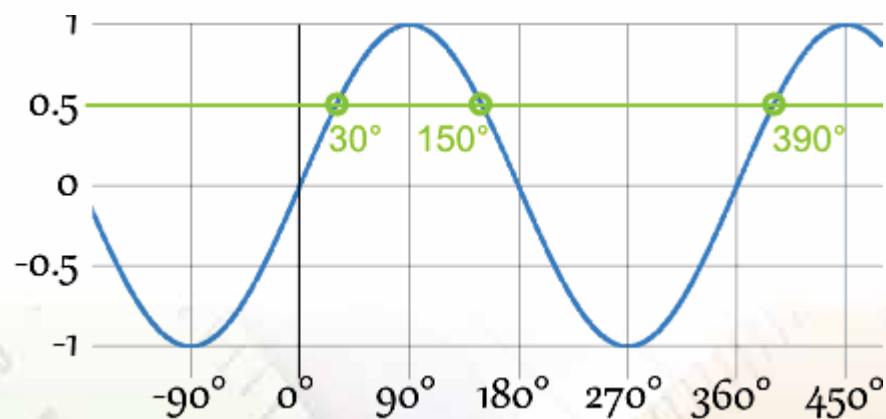


Some people like to remember the four letters **ASTC** by one of these:

- **All** Students **T**ake **C**hemistry
- **All** Students **T**ake **C**alculus
- **All** Same **T**ea **C**ups
- **Add** **S**ugar **T**o **C**offee

Two Values

Have a look at this graph of the Sine Function::



There are **two angles** (within the first 360°) that have the same value!

And this is also true for **Cosine** and **Tangent**.

The trouble is: **Your calculator will only give you one of those values...**

... but you can use these rules to find the other value:

	First value	Second Value
Sine	θ	$180^\circ - \theta$
Cosine	θ	$360^\circ - \theta$
Tangent	θ	$\theta - 180^\circ$

And if any angle is less than 0° , then add 360°

Solving Trigonometric equations

We can now solve equations for angles between 0° and 360° (using Inverse Sine , Cosine and Tangent)

To solve a trigonometric equations, use standard algebraic techniques such as collecting like terms and factoring. Your preliminary goal in solving a trigonometric equation is to isolate the trigonometric function in the equation.

For example, to solve the equation $2 \sin x = 1$, divide each side by 2 to obtain $\sin x = \frac{1}{2}$

Example 1: Solve $\sin \theta = 0.5$

We get the first solution from the calculator $= \sin^{-1}(0.5) = 30^\circ$ (it is in Quadrant I)

The other solution is $180^\circ - 30^\circ = 150^\circ$ (Quadrant II)

Example 2: Solve $\tan \theta = -1.3$

We get the first solution from the calculator $= \tan^{-1}(-1.3) = -52.4^\circ$

This is less than 0° , so we add 360° : $-52.4^\circ + 360^\circ = 307.6^\circ$ (Quadrant IV)

The other solution is $307.6^\circ - 180^\circ = 127.6^\circ$ (Quadrant II)

Example 3: Solve $\cos \theta = -0.85$

We get the first solution from the calculator $= \cos^{-1}(-0.85) = 148.2^\circ$ (Quadrant II)

The other solution is $360^\circ - 148.2^\circ = 211.8^\circ$ (Quadrant III)

Example 4: Solve $\sin(x + 10) = 0.5$ for $0 \leq x \leq 360$

$$(x + 10) = 30^\circ$$

There are two values for $(x + 10)$ that satisfy the equation; they are 30° and 150° .

$$x + 10 = 30 \quad \text{or} \quad x + 10 = 150$$

$$x = 20 \quad \text{or} \quad x = 140$$

Example 5 : Solve $1 + 2 \sin x = 2$ for $0 \leq x \leq 360$

$$2 \sin x = 1$$

$$\sin x = 0.5$$

From example 1,

$$x = 30^\circ$$

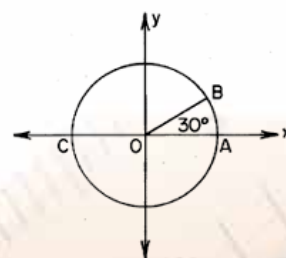
There are two values for x that satisfy the equation; they are 30° and 150°

$$x = 30 \text{ or } x = 150.$$

In the unit circle shown at the right, what are the coordinates of B ?

$\sin(x + 20^\circ) = \cos x$, the value of x is

What is the amplitude of the graph of the equation?



Important Angles: 30°, 45° and 60° (*Exact Values*)

You should try to **remember** sin, cos and tan for the angles 30°, 45° and 60°.

Yes, yes, it is a pain to have to remember things, but it will make life easier when you know them, not just in exams, but other times when you need to do quick estimates, etc.

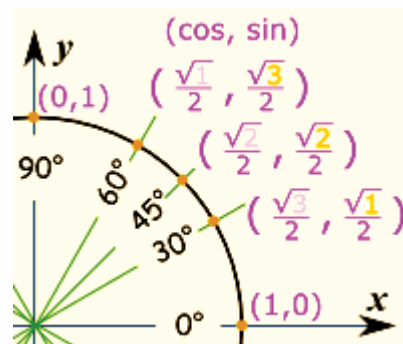
These are the values you should remember!

How To Remember?

$$\sin(30^\circ) = \frac{\sqrt{1}}{2} = \frac{1}{2} \quad (\text{because } \sqrt{1} = 1)$$

$$\sin(45^\circ) = \frac{\sqrt{2}}{2}$$

$$\sin(60^\circ) = \frac{\sqrt{3}}{2}$$



And cos goes "**3, 2, 1**"

$$\cos(30^\circ) = \frac{\sqrt{3}}{2}$$

$$\cos(45^\circ) = \frac{\sqrt{2}}{2}$$

$$\cos(60^\circ) = \frac{\sqrt{1}}{2} = \frac{1}{2} \quad (\text{because } \sqrt{1} = 1)$$

Just 3 Numbers

In fact, knowing 3 numbers is enough: $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$

EXERCISE 6.2

1. Determine the amplitude and period of the following trigonometric equations.

[a] $y = 3\sin 3x$

[b] $y = \frac{1}{2}\cos 2x$

[c] $y = \tan 2\theta$

[d] $y = -2\sin \frac{1}{2}\theta$

[e] $y = -\cos \theta$

[f] $y = 3\tan 4x$

2. On the same set of axes from 0 to 360° sketch the graph of

$$y = 2\cos\left(\frac{1}{2}x\right) \text{ and } y = \sin(2x)$$

3. Sketch the graph of $y = \sin\left(\frac{1}{2}x\right)$

4. Sketch the following graphs in the domain $0 \leq x \leq 360^\circ$

[a] $y = 3\sin 2x$

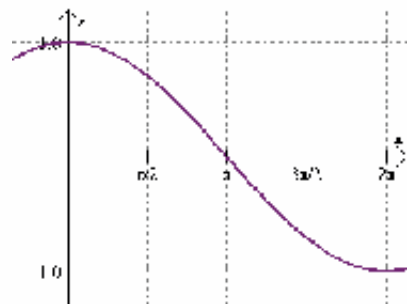
[b] $y = 2\cos 3x$

[c] $y = \frac{1}{2}\cos 2x$

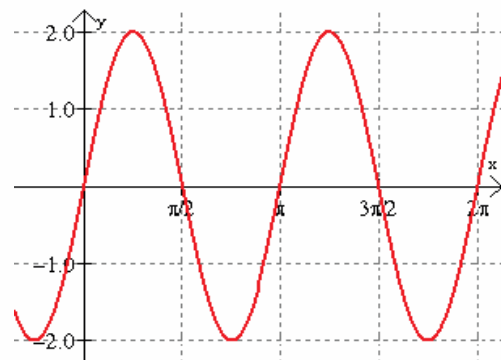
[d] $y = -3\sin 2x$

5. Determine the equation of the following graphs.

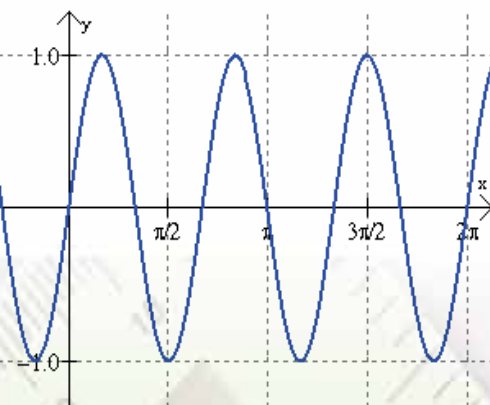
[a]



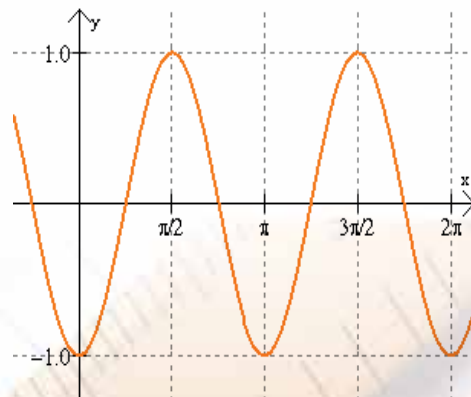
[b]



[c]



[d]



STRAND 7

STATISTICS

LEARNING OUTCOMES

Even though you may not have realized it, you probably have made some statistical statements in your everyday conversation or thinking. Statements like "I sleep for about eight hours per night on average" and "You are more likely to pass the exam if you start preparing earlier" are actually statistical in nature.

- Able to understand terms like frequency ,data , sample , tally
- Able to collect grouped and ungrouped data in different forms
- Able to find mean , mode ,median and quartiles for ungrouped data
- Able to interpret frequency graph

Statistics is a discipline which is concerned with:

- designing experiments and data collection,
- organising and summarizing information to aid understanding,
- analysing and drawing conclusions from data, and
- estimating the present or predicting the future.

In making predictions, Statistics uses the companion subject of Probability, which models chance mathematically and enables calculations of chance in complicated cases.

Today, statistics has become an important tool in the work of many academic disciplines such as medicine, psychology, education, sociology, engineering and physics, to name a few. Statistics is also important in many aspects of society such as business, industry and government. Because of the increasing use of statistics in so many areas of our lives, it has become very desirable to understand and practise statistical thinking. This is important even if you do not use statistical methods directly.

History of Statistics

The word statistics has been derived from Latin word "**Status**" or the Italian word "**Statista**", meaning "**Political State**" or a Government. Shakespeare used the word Statist in his drama Hamlet (1602). In the past, statistics was used by rulers. The application of statistics was very limited but rulers and kings needed information about lands, agriculture, commerce, population of their states, to assess their military potential, their wealth, taxation and other aspects of government.

Gottfried Achenwall used the word statistik at a German University in 1749 which meant the political science of different countries. In 1771 W. Hooper (Englishman) used the word statistics in his translation of Elements of Universal Erudition written by Baron B.F. Bieford. In his book, statistics has been defined as the science that teaches us what the political arrangement of all the modern states of the known world is. There is a big gap between old statistics and modern statistics, but old statistics is also used as a part of present day statistics.

During the 18th century, English writers extensively used the word statistics in their works, so statistics has developed gradually during the last few centuries. A lot of work has been done towards the end of the nineteenth century.

At the beginning of the 20th century, William S Gosset developed methods for decision making based on small sets of data. Several other statisticians have been active in developing new methods, theories and application of statistics. The availability of electronics and computers is certainly a major factor in the development of modern statistics.

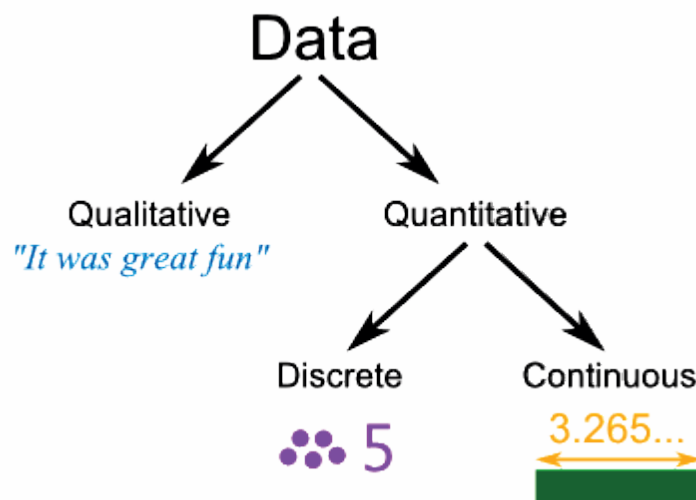
Data

Data is a collection of facts, such as values or measurements. It can be numbers, words, measurements, observations or even just descriptions of things.

Qualitative vs Quantitative Data

Data can be qualitative or quantitative.

- **Qualitative data** is descriptive information (it *describes* something). Also known as *Categorical data*
- **Quantitative data** is numerical information (numbers).



Quantitative data can be Discrete or Continuous:

- **Discrete data** can only integer values (like whole numbers)
- **Continuous data** can take any value (within a range or from the set of real numbers)

Simply put: **Discrete data** is counted, **Continuous data** is measured

Example: What do we know about Kuli the Dog?



Qualitative:

He is brown and black
He has long hair
He has lots of energy

Quantitative:

Discrete

He has 4 legs
He has 2 brothers

Continuous

He weighs 25.5 kg
He is 565 mm tall

To help you remember think "**Quantitative** is about **Quantity**"

Collecting Data

Data can be collected in many ways. The simplest way is **direct observation**.

Example:

You want to find how many cars pass by a certain point on a road in a 20-minute interval.

Stand at that point on the road, and count the cars that pass by in that interval.

You collect data by doing a **Survey**

Census or Sample

A **Census** is when you collect data for **every** member of the group (the whole **population**).

A **Sample** is when you collect data just for **selected members or elements** of the group.

Example:

There are 120 people in your local football club.

You can ask everyone (all 120) what their age is. That is a **census**.

Or you could just choose the people that are there this afternoon. That is a **sample**.

A census is accurate, but hard to do. A sample is not as accurate, but may be good enough, and is a lot easier.

Data or Datum

Singular form is "datum", so we would say "that datum **is** very high".

"Data" is the plural of datum so we can say "the data **are** available", but it is also a **collection** of facts, so "the data **is** available" is fine too.

How to Do a Survey

Turn on the television, radio or open a newspaper and you will often see the results from a survey.

Gathering information is an important way to help people **make decisions** about topics of interest.

Surveys can help decide what needs changing, where money should be spent, what products to purchase, what problems there might be, or lots of other questions you may have at any time. The best part about surveys is that they can be used to answer any question about any topic.

You can survey **people** (through questionnaires, opinion polls, etc) or **things** (like pollution levels in a river, or traffic flow)

Four Steps

Here are four steps to a successful survey:

- create the questions
- ask the questions
- tally the results
- present the results

Finally make relevant and appropriate conclusion

Let us look at those steps in more detail...

Step One: Create the Questions

The first thing is to decide

What questions do you want answered.

Sometimes these may be simple questions like:

- *"What is your favourite colour?"*

Other times the questions may be quite complex such as:

- *"Which roads have the worst traffic conditions?"*

Simple Surveys

If you are doing a simple survey, you could use tally marks to represent each person's answer:

Yellow		4
Red		5
Blue		6
Green		1
Pink		4

Sometimes, it is helpful to be creative in how people can respond. It makes it more fun for both you and your respondents (the people answering the question).

Example: What is your favourite colour?

Have them write down their favourite colour on a piece of paper and drop it in a fish bowl. Then, put all of the pieces of paper into piles and count them.

To help you make a good Questionnaire read our page Survey Questions.

Step Two: Asking the Questions

Now you have your questions, **go out and ask them!** But who to ask?

If you survey a small group you can ask everybody (called a **Census**)

If you want to survey a large group, you may not be able to ask everybody so you should ask a smaller number of the population (called a **Sample**)

If you are **sampling** you should be careful who you ask.

To be a good sample, each person should be chosen **randomly**



- If you only ask people who look friendly, you will only know what friendly people think!

If you went to the swimming pool and asked people "Can you swim?" you will get a **biased** answer ... maybe even 100% will say "Yes".

Note: Surveys where people are asked to ring a number to vote are not very accurate, because only **certain types of people** actually ring up!

So be careful not to bias your survey. Try to choose randomly.

Example: You want to know the favourite colours for people at your school, but don't have the time to ask everyone.

Solution: Choose 50 people at random:

- stand at the gate and choose "the next person to arrive" each time
- *or* choose people randomly from a list and then go and find them!
- *or* you could choose every 5th person

Your results will **hopefully** be nearly as good as if you asked everyone.

If you choose a person and they do not want to answer, just record "no answer" on the survey form and mention how many people did not answer in your report.

After completing a sampling survey you can use the information to make a **prediction** as to how the rest of the population would respond.

The more people you have asked, the better your result will be.

Example: Nationwide opinion polls survey up to 2,000 people and the results are nearly as good (within about 1%) as asking everyone.

Step Three: Tally the Results

Now you have finished asking questions it is time to tally the results.

Tally means to **add up**. This usually involves lots of paperwork and computer work (spread sheets are useful!)

Example: For "favourite colours of my class" you can simply write tally marks like this (every fifth mark crosses the previous 4 marks, so you can easily see groups of 5):

Yellow		4
Red		5
Blue		6
Green		1
Pink		4

Step Four: Presenting the Results

Now you have your results, you will want to show them to other people in the best possible way. Results can be presented via **tables** and **graphs**.

Tables

Sometimes, you can simply report the information in a table.

A table is a very simple way to show others the results. A table should have a title, so those looking at it understand what results the table shows:

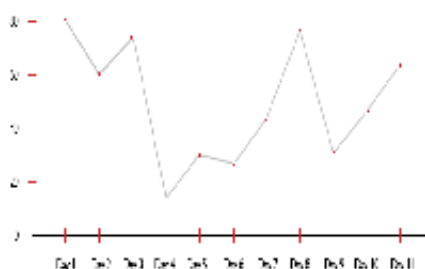
<i>Table: The Favourite Colours of My Class</i>				
Yellow	Red	Blue	Green	Pink
4	5	6	1	4

Types of Graphs

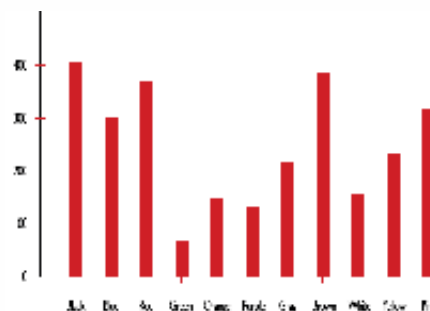
Graphs are used in a variety of ways, and in almost every industry, such as medicine, agriculture, engineering, optimization, mathematics, and education. Graphs help us to visualize data. What we see on the graph may help us to interpret and analyze the data.

The end result is usually a better perspective that may not have been possible just by looking at a set of numbers or tables. Common graphs used in mathematics are shown below.

Line Graphs



Bar Graphs



Pictographs



Pie Chart



Every graph must:

- have a title
- be labelled properly
- (axes should be) equally/uniformly distributed or scaled
-

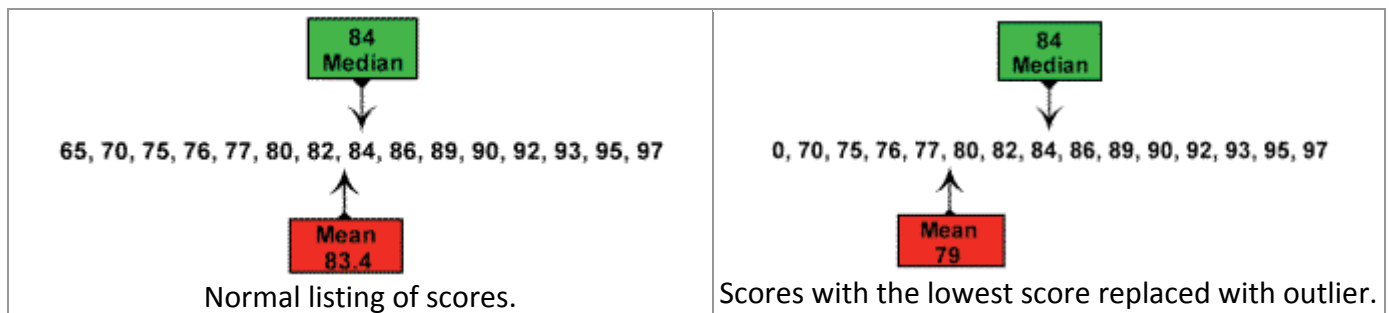
Note: The width of bars in a Bar graph or Histogram must be equal

Measures of Central Tendency

The term "**measures of central tendency**" refers to finding the **mean, median and mode**.

Mean:	Average. The sum of a set of data divided by the number of data. (Do not round your answer unless directed to do so.)
Median:	The middle value, or the mean of the middle two values, when the data is arranged in numerical order either ascending or descending. Think of a "median" being in the middle of a highway.
Mode:	The value (number) that appears the most . It is possible to have more than one mode, and it is possible to have no mode. If there is no mode-write "no mode", do not write zero (0) .

Consider these sets of test score values:



The two sets of scores above are identical except for the first score. The set on the left shows the actual scores. The set on the right shows what would happen if one of the scores was WAY out of range in regard to the other scores. Such a term is called an outlier.

With the outlier, the mean changed.

With the outlier, the median did NOT change.

In both cases, there is **no mode**.

How to decide which measure of central tendency to use

MEAN

Use the mean to describe the middle of a set of data that *does not* have an outlier.

Advantages:

- Most popular measure in fields such as business, engineering and computer science.
- It is unique - there is only one answer.
- Useful when comparing sets of data.

Disadvantages:

- Affected by extreme values (outliers)

MEDIAN

Use the median to describe the middle of a set of data that *does* have an outlier.

Advantages:

- Extreme values (outliers) do not affect the median as strongly as they do the mean.
- Useful when comparing sets of data.
- It is unique - there is only one answer.

Disadvantages:

- Not as popular as mean.

MODE

Use the mode when the data is non-numeric or when asked to choose the most popular item.

Advantages:

- Extreme values (outliers) do not affect the mode.

Disadvantages:

- Not as popular as mean and median.
- Not necessarily unique - may be more than one answer
- When no values repeat in the data set, the mode is every value and is useless.
- When there is more than one mode, it is difficult to interpret and/or compare.

What will happen to the measures of central tendency if we add the same amount to all data values, or multiply each data value by the same amount?

	Data	Mean	Mode	Median
Original Data	6, 7, 8, 10, 12, 14, 14, 15, 16, 20	12.2	14	13
Add 3 to each value	9, 10, 11, 13, 15, 17, 17, 18, 19, 23	15.2	17	16
Multiply 2 to each Value	12, 14, 16, 20, 24, 28, 28, 30, 32, 40	24.4	28	26

When added: Since all values are **shifted** the same amount, the measures of central tendency all shifted by the same amount. If you add 3 to each data value, you will add 3 to the mean, mode and median.

When multiplied: Since all values are affected by the same multiplicative values, the measures of central tendency will feel the same effect. If you multiply each data value by 2, you will multiply the mean, mode and median by 2.

Example 1

Find the mean, median and mode for the following data: 5, 15, 10, 15, 5, 10, 10, 20, 25, 15.

(You will need to organize the data.)

5, 5, 10, 10, 10, 15, 15, 15, 20, 25

Mean:

$$\frac{\text{sum of data}}{\text{number of data}} = \frac{130}{10} = 13$$

Median: 5, 5, 10, 10, **10, 15**, 15, 15, 20, 25

Listing the data in order is the easiest way to find the median.
The numbers **10 and 15 both** fall in the middle.

Average these two numbers to get the median. $\frac{10 + 15}{2} = 12.5$

Mode: Two numbers appear **most often: 10 and 15**. There are three 10's and three 15's.
Hence, there are two modes – **10 and 15**.

Example 2

For what value of x will 8 and x have the same mean (average) as 27 and 5?

First, find the mean of 27 and 5:

$$\frac{27 + 5}{2} = 16$$

Now, find the x value, knowing that the average of x and 8 must be 16:

$$\frac{x + 8}{2} = 16$$

$$32 = x + 8 \quad \text{cross multiply}$$

$$\begin{array}{r} -8 \\ 32 = x + 8 \\ \hline \end{array}$$

$$24 = x \quad \text{and solve}$$

Example 3:

On her first 5 Mathematics tests, Aliya received the following scores: **72, 86, 92, 63** and **77**. What test score must Aliya earn on her sixth test so that her average (mean score) for all six tests will be 80? .

Possible solution:

Set up an equation to represent the situation. Remember to use all 6 test scores:

$$\frac{72 + 86 + 92 + 63 + 77 + x}{6} = 80$$

cross multiply and solve:

$$(80)(6) = 390 + x$$

$$480 = 390 + x$$

$$\begin{array}{r} -390 \\ 480 = 390 + x \\ \hline \end{array}$$

$$90 = x$$

Aliya must get a 90 on the sixth test.

Example 4

The mean (average) weight of three dogs is 38 kg. One of the dogs, Sparky, weighs 46 kg. The other two dogs, Eddie and Sandy, have the same weight. Find Eddie's weight.

Let x = Eddie's weight (they weigh the same, so they are both represented by " x ".)

Let x = Sandy's weight

Average: sum of the data divided by the number of data.

$$\frac{x + x + 46}{3} = 38 \quad \text{cross multiply and solve}$$

3(dogs)

$$(38)(3) = 2x + 46$$

$$114 = 2x + 46$$

$$\frac{-46}{2} \quad \frac{-46}{2}$$

$$\frac{68}{2} = \frac{2x}{2}$$

$$34 = x$$

Eddie weighs 34 pounds.

Measures of Dispersion

Another way of examining single variable data is to look at how the data is spread out, or dispersed about the mean.

The smaller the values from these methods, the more consistent the data.

It is possible to get a sense of a data set's distribution by examining a five statistical summary, the (1) *minimum*, (2) *maximum*, (3) *median* (or *second quartile*), (4) the *lower quartile*, and (5) the *upper quartile*. Such information will show the extent to which the data is located near the median or near the extremes.

1 Range

The simplest of our methods for measuring dispersion is *range*. Range is the difference between the maximum value and the minimum value in the data set. While being simple to compute, the range is often unreliable as a measure of dispersion since it is based on only two values in the set.

12, 25, 27, 29, 36, 38, 40, 43, 50, 54, 62

$$\text{Range} = 62 - 12 = 50$$

2. Quartiles

We know that the **median** of a set of data separates the data into two equal parts. Data can be further separated into **quartiles**. Quartiles separate the original set of data into four equal parts. Each of these parts contains one-fourth of the data.

Quartiles are **percentiles** that divide the data into fourths.

The Lower(first) quartile is the middle (the median) of the lower half of the data. One-fourth of the data lies below the first quartile and three-fourths lies above. (the 25th percentile)	The second quartile is another name for the median of the entire set of data. Median of data set = second quartile of data set. (the 50th percentile)	The Upper (second) quartile is the middle (the median) of the upper half of the data. Three-fourths of the data lies below the third quartile and one-fourth lies above. (the 75th percentile)
---	--	--

Check out this five statistical summary for a set of tests scores.

minimum	first quartile	second quartile (median)	third quartile	maximum
65	70	80	90	100

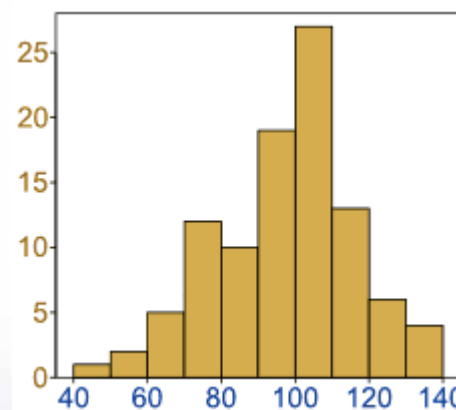
The **interquartile range (IQR)**, also called the mid spread or middle fifty, is the range between the third and first quartiles and is considered a more stable statistic than the total range. The IQR contains 50% of the data.

Histograms

A Histogram is a graphical display of data using bars of different heights.

It is similar to a Bar Chart, but a histogram groups numbers into **ranges**.

And you decide what ranges to use!



Example: Height of Trees

You measure the height of every tree in the garden in centimetres (cm)

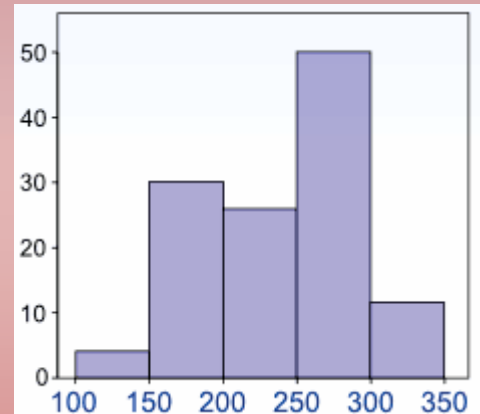
The heights vary from 100 cm to 340 cm.

You decide to put the results into groups of 50 cm:

- The **100 to just below 150 cm** range,
- The **150 to just below 200 cm** range,
- etc...

So a tree that is 260 cm tall is added to the "250-300" range.

And here is the result.



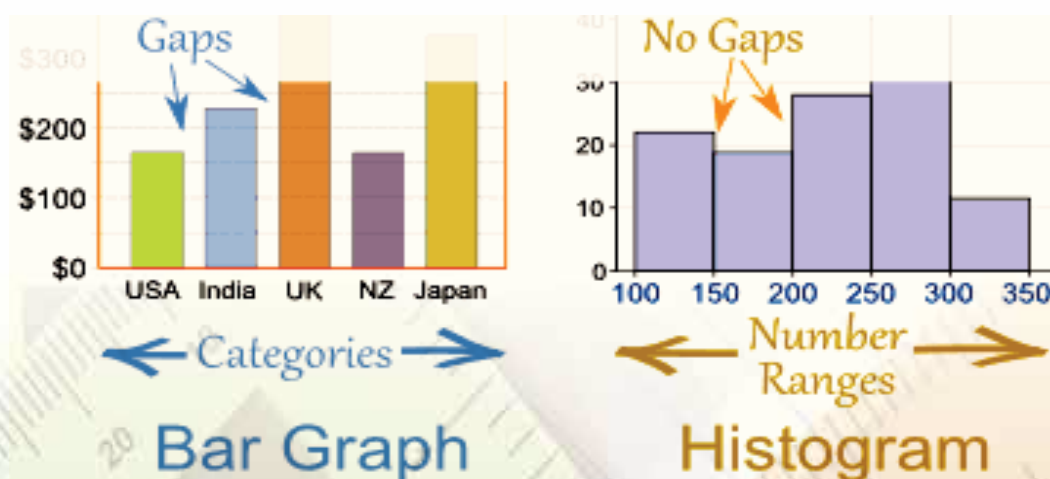
You can see (for example) that there are **30** trees from **150 cm to just below 200 cm** tall

The range of each bar is also called the **Class Interval**.

Histograms are a great way to show results of **continuous** data, such as:

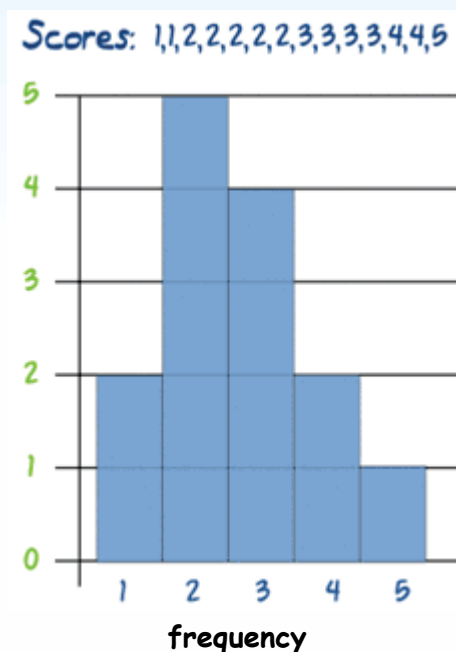
- weight
- height
- how much time
- etc.

But when the data is in **categories** (such as Country or Favourite Movie), we should use a Bar Chart.



Frequency Histogram

A Frequency Histogram is a special histogram that uses vertical columns to show frequencies (how many times each score occurs):



Cumulative Frequency

Cumulative means "how much so far".

Think of the word "accumulate" which means to gather together.

To have cumulative totals, just **add up the values as you go**.

The total of a frequency and all frequencies so far in a frequency distribution.

It is the 'running total' of frequencies.

Scores: 1,1,2,2,2,2,2,3,3,3,3,4,4,5

Score	Frequency	Cumulative Frequency
1	2	2
2	5	7
3	4	11
4	2	13
5	1	14

Cumulative Frequency for Score 3
is $2+5+4 = 11$

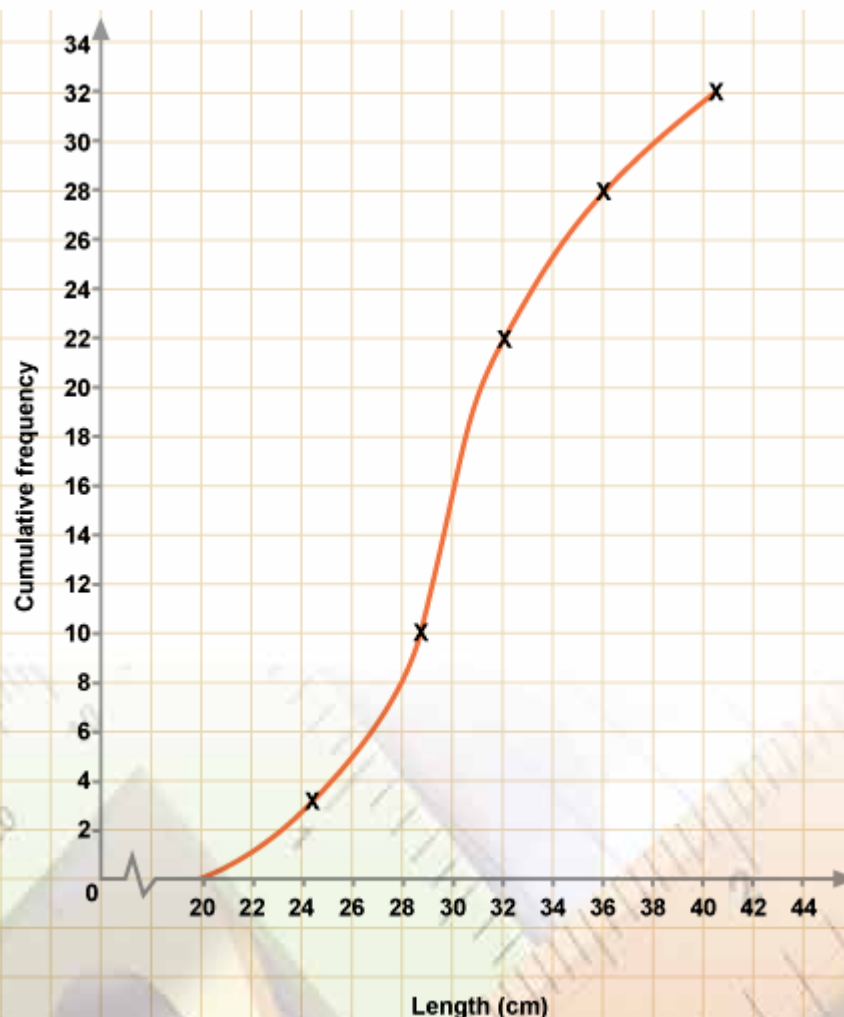
Drawing a cumulative frequency diagram

The table shows the lengths (in cm) of 32 cucumbers.

Before drawing the cumulative frequency diagram, we need to work out the cumulative frequencies. This is done by adding the frequencies in turn.

Length	Frequency	Cumulative Frequency
21-24	3	3
25-28	7	10 (= 3 + 7)
29-32	12	22 (= 3 + 7 + 12)
33-36	6	28 (= 3 + 7 + 12 + 6)
37-40	4	32 (= 3 + 7 + 12 + 6 + 4)

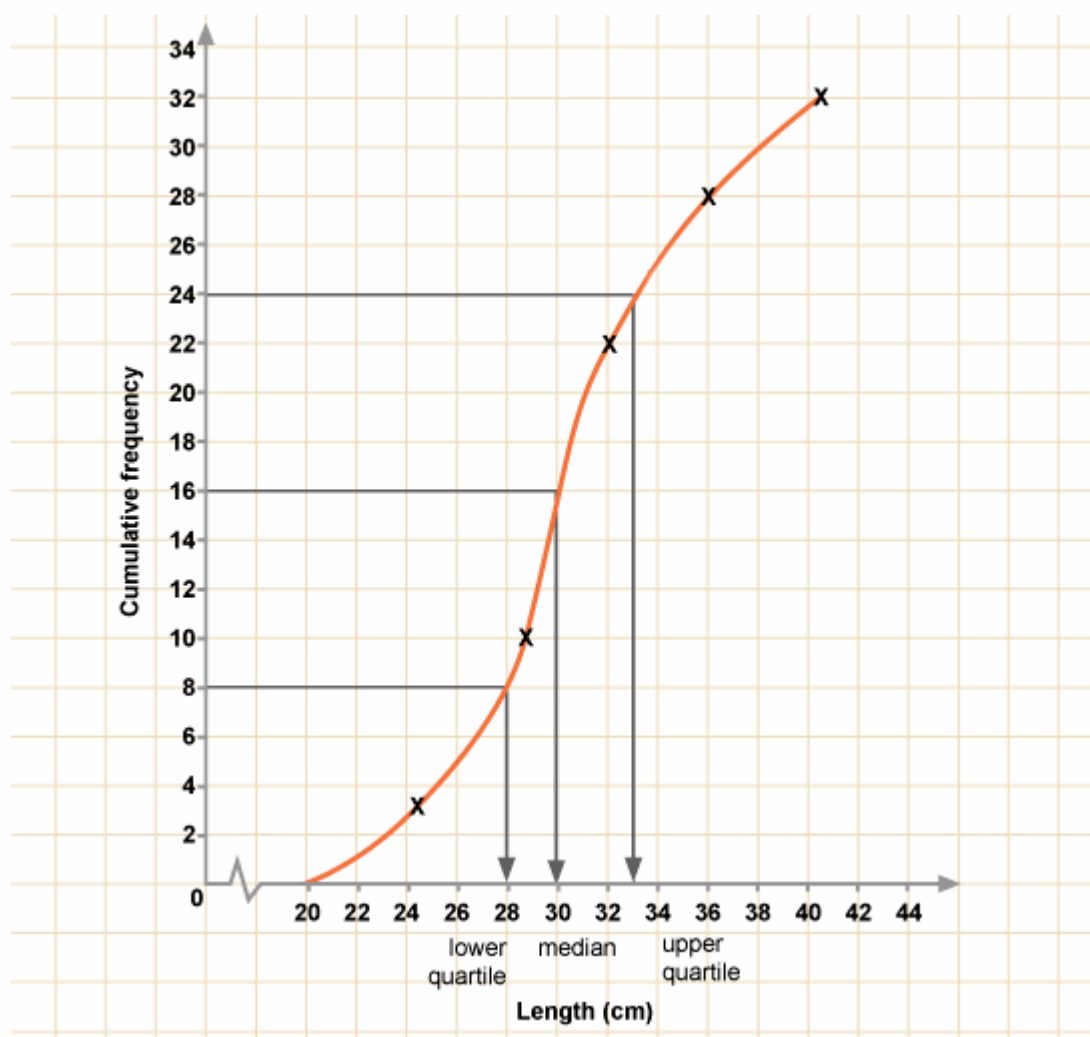
The points are plotted at the upper class boundary. In this example, the upper class boundaries are 24.5, 28.5, 32.5, 36.5 and 40.5. Cumulative frequency is plotted on the vertical axis.



Finding the median and quartiles

When looking at a cumulative frequency curve, you will need to know how to find its median, lower and upper quartiles, and the interquartile range.

By drawing horizontal lines to represent $\frac{1}{4}$ of the total frequency, $\frac{1}{2}$ of the total frequency and $\frac{3}{4}$ of the total frequency, we can read estimates of the lower quartile, median and upper quartile respectively from the horizontal axis.



Quartiles are associated with quarters. The **interquartile range** is the **difference** between the lower and upper quartile.

From these values, we can also estimate the interquartile range: $33 - 28 = 5$.

Remember to use the total frequency, **not** the maximum value, on the vertical axis. The values are always read from the horizontal axis.

EXERCISE 7.1

- The weekly salaries of six employees at McDonalds are \$140, \$220, \$90, \$180, \$140, \$200. For these six salaries, find: (a) the mean (b) the median (c) the mode.
- A storeowner kept a tally of the sizes of suits purchased in her store.
 - Which measure of central tendency should the storeowner use to describe the average size suit sold?
 - A tally was made of the number of times each colour of crayon was used by a kindergarten class. Which measure of central tendency should the teacher use to determine which colour is the favourite colour of her class?
- Test scores for a class of 20 students are as follows:
93, 84, 97, 98, 100, 78, 86, 100, 85, 92, 72, 55, 91, 90, 75, 94, 83, 60, 81, 95
 - Copy and complete the table shown at the left.

Test Scores	Frequency
91-100	
81-90	
71-80	
61-70	
51-60	

- Find the modal interval.

The "**modal interval**" is the interval containing the greatest frequency. It is not the mode.

- Find the interval that contains the median.

- The values of 11 houses on Brown Street are shown in the table.

- Find the mean value of these houses in dollars.
- Find the median value of these houses in dollars
- State which measure of central tendency, the mean or the median, best represents the values of these 11 houses. Justify your answer.

Value per House	Number of Houses
\$100,000	1
\$175,000	5
\$200,000	4
\$700,000	1

- Know and understand simple probability terms
- Draw simple probability tree
- List sample space
- Calculate simple probabilities

Brief History of Probability

Concepts of probability have been around for

thousands of years, but probability theory did not arise as a branch of mathematics until the mid-seventeenth century. During the fifteenth century several probability works emerged. Calculations of probabilities became more noticeable during this time period even though mathematicians in Italy and France remained unfamiliar with these calculation methods (David, 1962). In 1494, Fra Luca Paccioli wrote the first printed work on probability, *Summa de arithmetica, geometria, proportioni e proportionalita* (David, 1962). In 1550, Geronimo Cardano inspired by the *Summa* wrote a book about games of chance called *Liber de Ludo Aleae* which means A Book on Games of Chance (David, 1962).

In the mid-seventeenth century, a simple question directed to Blaise Pascal by a nobleman sparked the birth of probability theory, as we know it today. Chevalier de Méré gambled frequently to increase his wealth. He bet on a roll of a die that at least one 6 would appear during a total of four rolls. From past experience, he knew that he was more successful than not with this game of chance. Tired of his approach, he decided to change the game. He bet that he would get a total of 12, or a double 6, on twenty-four rolls of two dice. Soon he realized that his old approach to the game resulted in more money. He asked his friend Blaise Pascal why his new approach was not as profitable. Pascal worked through the problem and found that the probability of winning using the new approach was only 49.1 percent compared to 51.8 percent using the old approach (see [detailed calculations](#)) (Smith, 1996).

This problem proposed by Chevalier de Méré is said to be the start of famous correspondence between Pascal and Pierre de Fermat. They continued to exchange their thoughts on mathematical principles and problems through a series of letters. Historians think that the first letters written were associated with the above problem and other problems dealing with probability theory. Therefore, Pascal and Fermat are the mathematicians credited with the founding of probability theory (David, 1962).

The topic of probability is seen in many facets of the modern world. The theory of probability is not just taught in mathematics courses, but can be seen in practical fields, such as insurance, industrial quality control, study of genetics, quantum mechanics, and the kinetic theory of gases (Simmons, 1992).

Probability is a branch of mathematics that deals with calculating the likelihood of a given event's occurrence, which is expressed as a number between 1 and 0. An event with a probability of 1 can be considered a certainty: for example, the probability of a coin toss resulting in either "heads" or "tails" is 1, because there are no other options, assuming the coin lands flat. An event with a probability of .5 can be considered to have equal odds of occurring or not occurring: for example, the probability of a coin toss resulting in "heads" is .5, because the toss is equally as likely to result in "tails." An event with a probability of 0 can be considered an impossibility: for example, the probability that the coin will land (flat) without either side facing up is 0, because either "heads" or "tails" must be facing up

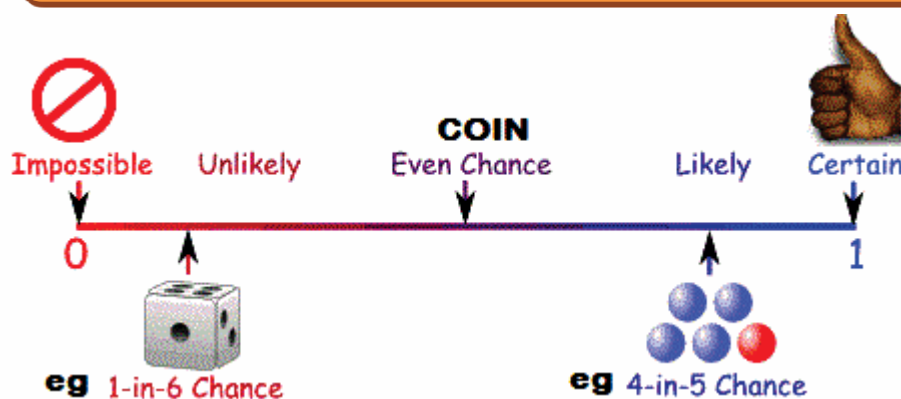
How likely something is to happen.

Many events cannot be predicted with total certainty.

The best we can say is how **likely** they are to happen, using the idea of probability.

In general:

$$\text{Probability of an event happening} = \frac{\text{Number of ways it can happen}}{\text{Total number of outcomes}}$$



Probability is **always** between 0 and 1

Sample Space: the set of all the possible outcomes of an experiment

There are 52 cards in a deck (not including Jokers)

So the **Sample Space** is all 52 possible cards: {Ace of Hearts, 2 of Hearts, etc... }

The Sample Space is made up of **Sample Points**.

Sample Point: just one of the possible outcomes

Event: a single result of an experiment

Example of Events:

- Getting a Tail when tossing a coin is an event
- Rolling a “5” is an event.

An event can include one or more possible outcomes:

- Choosing a “King” from a deck of cards (any of the 4 Kings) **is** an event
- Rolling an “even number” (2, 4 or 6) is also an event

Activity	Sample Space
Rolling a die 	There will be 6 outcomes in the sample space: {1, 2, 3, 4, 5, 6}
Tossing a coin 	There will be 2 outcomes in the sample space: { Heads, Tails }
Drawing a card from a standard deck 	There will be 52 cards in the sample space: {Spades: 2,3,4,5,6,7,8,9,10, ace, jack queen, king, Clubs: 2,3,4,5,6,7,8,9,10, ace, jack, queen, king, Diamonds: 2,3,4,5,6,7,8,9,10, ace, jack, queen, king, Hearts: 2,3,4,5,6,7,8,9,10, ace, jack, queen, king}

Drawing one marble from the bottle 	There will be 8 marbles in the sample space: {blue marble, blue marble, blue marble, blue marble, blue marble, red marble, red marble, red marble}
Rolling a pair of dice 	There will be 36 outcomes in the sample space: {(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (2,1) (2,2) (2,3) (2,4) (2,5) (2,6) (3,1) (3,2) (3,3) (3,4) (3,5) (3,6) (4,1) (4,2) (4,3) (4,4) (4,5) (4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

As you can see, sample spaces can become very large.

When determining a sample space, be sure to consider ALL possibilities. Oftentimes this can be a difficult task.

Remember: A sample space is the set of ALL possible outcomes.

When attempting to determine a **sample space** (the **possible outcomes** from an experiment), it is often helpful to draw a diagram which illustrates how to arrive at the answer.

One such diagram is a **tree diagram**.

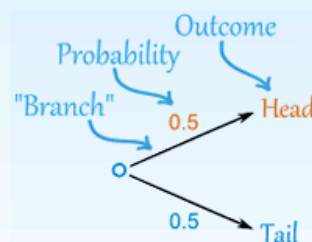
Probability Tree Diagrams

Calculating probabilities can be hard, sometimes we add them, sometimes we multiply them, and often it is hard to figure out what to do ... **tree diagrams to the rescue!**

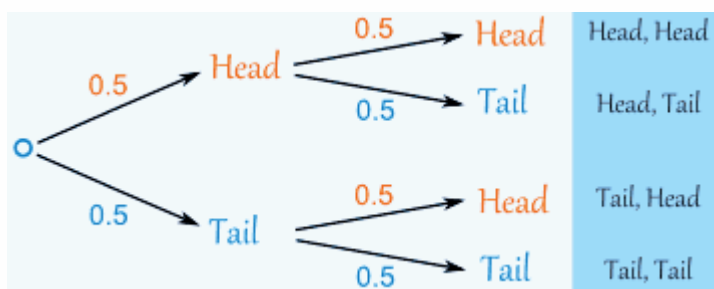
Here is a tree diagram for the toss of a coin:

There are two "branches" (Heads and Tails)

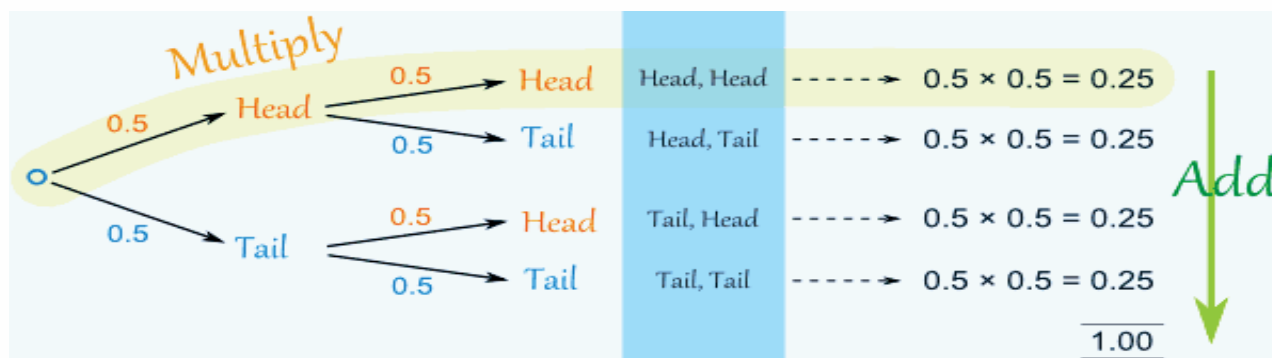
- The probability of each branch is written on the branch
- The outcome is written at the end of the branch



We can extend the tree diagram to two tosses of a coin:



How do we calculate the overall probabilities?



- We **multiply** probabilities **along the branches**
- We **add** probabilities down **columns**

Now we can see such things as:

- The probability of "Head, Head" is $0.5 \times 0.5 = \mathbf{0.25}$
- All probabilities add to **1.0** (which is always a good check)
- The probability of getting at least one Head from two tosses is $0.25 + 0.25 + 0.25 = \mathbf{0.75}$
- ... and more

In addition to helping determine the number of outcomes in a sample space, the tree diagram can be used to determine the probability of individual outcomes within the sample space.

The **probability** of any outcome in the sample space is the product (multiply) of all possibilities along the path that represents that outcome on the tree diagram.

Throwing Two Dice and Adding the Scores.

Question: If you throw 2 dice together and add the two scores:

- 1. What is the **least** possible total score?
- 2. What is the **greatest** possible total score?
- 3. What do you think is the **most likely** total score?

The first two questions are quite easy to answer:

- 1. The **least** possible total score must be $1 + 1 = 2$
- 2. The **greatest** possible total score must be $6 + 6 = 12$
- 3. The **most likely** total score is ... ???

Are they all just as likely? Or will some happen more often?

To help answer the third question let us try an experiment.

The Experiment

Here is a table of all possible outcomes, and the totals

		Score on One Die					
		1	2	3	4	5	6
Score on the Other Die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

What is the most likely total score?

- 7 has the highest bar, so 7 is the most likely total score.

Example: Probability of a total of 2

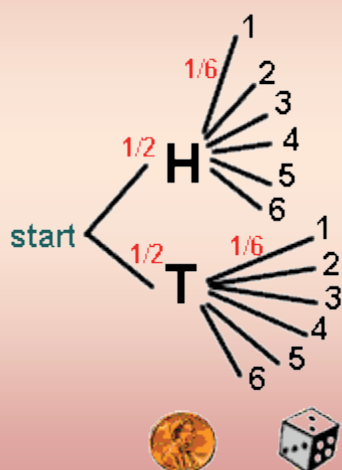
- We know there are 36 possible outcomes.
- And there is only 1 way to get a total score of 2.
- So the probability of getting 2 is:

$$\text{Probability of a total of 2} = \frac{1}{36}$$

Example 1 :

Show the sample space for tossing one coin and rolling one die.

(H = heads, T = tails)



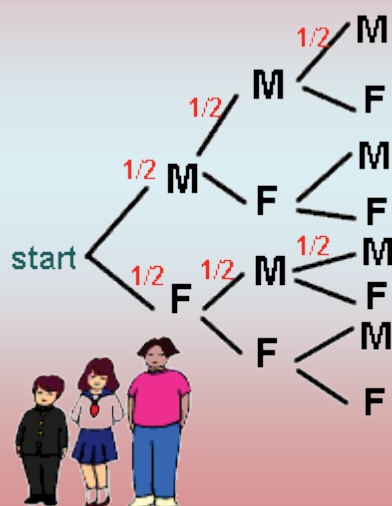
By following the different paths in the tree diagram, we can arrive at the sample space.

Sample space:

{ H1, H2, H3, H4, H5, H6,
T1, T2, T3, T4, T5, T6 }

The **probability** of each of these outcomes is
 $1/2 \cdot 1/6 = 1/12$

Example 2: A family has three children. How many outcomes are in the sample space that indicates the sex of the children? Assume that the probability of male (M) and the probability of female (F) are each $1/2$.



Sample space:

{ MMM
MMF
MFM
MFF
FMM
FMF
FFM
FFF }

There are 8 outcomes in the sample space.

The **probability** of each outcome is
 $1/2 \cdot 1/2 \cdot 1/2 = 1/8$.

EXERCISE

8.1

- How many elements are in the sample space of rolling one die? List the sample space.
- How many elements are in the sample space of tossing 3 coins? List the sample space.



- Alarm clocks are sold in blue or pink with either digital or standard displays. How many different arrangements of alarm clocks are possible? List the sample space.



- There are 2 entry doors and 3 staircases in a two-storey school building. How many ways are there to enter the building and go to the first floor? List the sample space.

5. You want to travel from Suva to Ba. You can either travel via the Kings Road or the Queens Highway, using a bus, taxi, private car or rental. By how many possible ways could you travel? List the sample space.

6. A die is thrown once. What is the probability that the score is a factor of 6?

7. The diagram shows a spinner made up of a piece of card in the shape of a regular pentagon, with a toothpick pushed through its centre. The five triangles are numbered 1 to 5.



The spinner is spun until it lands on one of the five edges of the pentagon. What is the probability that the number it lands on is odd?

8. Each of the letters of the word MATHEMATICS are written on separate pieces of paper that are then folded, put in a hat, and mixed thoroughly.

One piece of paper is chosen (without looking) from the hat. What is the probability it is an 'l'?

- 9.. A fair coin is tossed three times. What is the probability of obtaining one Head and two Tails? (A fair coin is one that is not loaded, so there is an equal chance of it landing Heads up or Tails up.)

- 10 A committee of three is chosen from five councilors – Ansaar , Shalen, Mere, Jone and Aarti.

What is the probability that Ramesh is on the committee?

- 11 A special die is made in the shape of an icosahedron, and its faces are numbered with the numbers 1 to 20.



When the die is thrown there is an equal chance of any face landing uppermost.

If the die is thrown once, what is the probability that the face that lands uppermost has a number that is a factor of 20?

- 12 Two cubical dice are thrown and their scores added together.

If $X = \text{"The sum of the scores on the two dice"}$, what is:

- $P(X = 7 \text{ or } 8)$
- $P(X \text{ is divisible by } 4)$



First die Second die

STRAND 9

INTRODUCTION TO CALCULUS

LEARNING OUTCOMES

- Know and understand differentiation and integration
- Understand differentiation with reference to graph
- Differentiate and integrate simple algebraic expressions.

Calculus is all about **changes**.

The word Calculus comes from Latin meaning "**small stone**". Because it is like understanding something by looking at small pieces.

It is the branch of mathematics that deals with the finding and properties of derivatives and integrals of functions, by methods originally based on the summation of infinitesimal differences. The two main types are *differential calculus* and *integral calculus*.

Differential Calculus cuts something into small pieces to find how it changes.

Integral Calculus joins (integrates) the small pieces together to find how much there is.

Differential Calculus and Integral Calculus are inverses of each other, just like multiplication and division are inverses.

Brief History of Calculus

Calculus was created by Isaac Newton, a British scientist, as well as Gottfried Leibniz, a self-taught German mathematician, in the 17th century.. Newton discovered the inverse relationship between the derivative (slope of a curve) and the integral (the area beneath it), which deemed him as the creator of calculus. Thereafter, calculus was actively used to solve the major scientific dilemmas of the time, such as:

- *calculating the slope of the tangent line to a curve at any point along its length*
- *determining the velocity and acceleration of an object given a function describing its position, and designing such a position function given the object's velocity or acceleration*
- *calculating arc lengths and the volume and surface area of solids*
- *calculating the relative and absolute extrema of objects, especially projectiles*

For Newton, the applications for calculus were geometrical and related to the physical world - such as describing the orbit of the planets around the sun. For Leibniz, calculus was more about analysis of change in graphs. Leibniz's work was just as important as Newton's, and many of his notations are used today, such as the notations for taking the derivative and the integral.

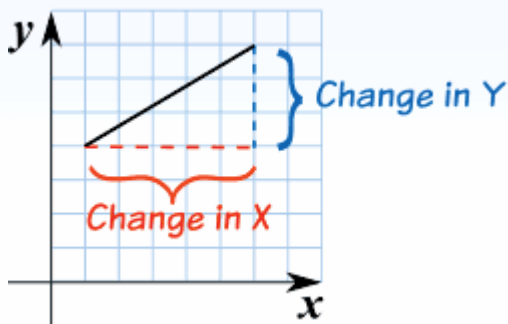
DIFFERENTIATION - Introduction to Derivatives

The **derivative** of a function f at a given value x is the instantaneous rate of change of the function f or the **slope/gradient of the tangent** at x .

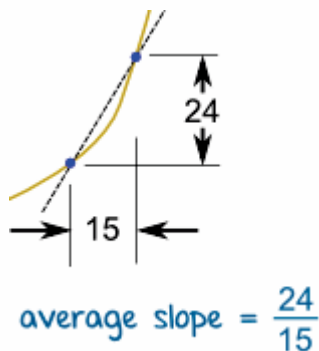
The rate of change of a function at a specific value of x .

It is all about slope or gradient!

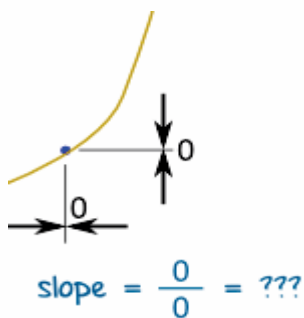
$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}}$$



We can find an **average** slope between two points.



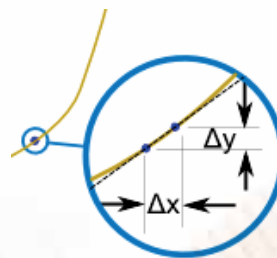
But how do we find the slope **at a point**?



There is nothing to measure!

But with derivatives we use a small difference then have

it **shrink towards zero** (approach 0).

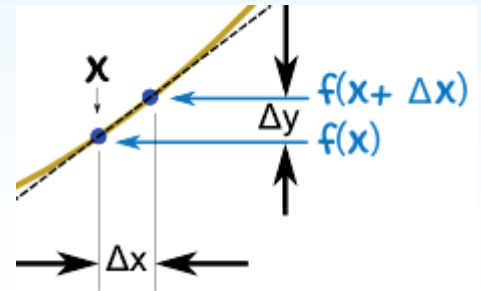


Let us find a Derivative!

We will use the slope formula:

$$\text{Slope} = \frac{\text{Change in } Y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$$

to find the derivative of a function $y = f(x)$



Follow these steps:

- Fill in this slope formula: $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$
- Simplify it as best as possible
- Then make Δx shrink towards zero.

Here we go:

Example 1: the function $f(x) = x^2$

We know $f(x) = x^2$, and can calculate $f(x+\Delta x)$:

$$\text{Start with: } f(x+\Delta x) = (x+\Delta x)^2$$

$$\text{Expand } (x + \Delta x)^2: f(x+\Delta x) = x^2 + 2x \Delta x + (\Delta x)^2$$

Start with the slope formula:

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Put in $f(x+\Delta x)$ and $f(x)$:

$$\frac{x^2 + 2x \Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

Simplify (x^2 and $-x^2$ cancel):

$$= \frac{2x \Delta x + (\Delta x)^2}{\Delta x}$$

Simplify more (divide through by Δx):

$$= 2x + \Delta x$$

And then as Δx heads towards 0 we get: $= 2x + 0$

Result: The derivative of x^2 is $2x$

We write **dx** instead of "**Δx heads towards 0**", so "the derivative of" is commonly written

$$\frac{d}{dx} (x^2) = 2x$$

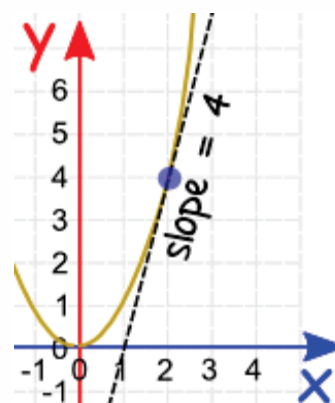
"The derivative of x^2 equals $2x$ "
or simply "d/ dx of x^2 equals $2x$ "

What does $\frac{d}{dx}(x^2) = 2x$ mean?

It means that, for the function x^2 , the slope or "rate of change" at any

Point is $2x$. So when $x = 2$ the slope is $2x = 4$, as shown here:

Or when $x = 5$ the slope is $2x = 10$, and so on.



Note: sometimes $f'(x)$ is also used for "the derivative of":

$f'(x) = 2x \Rightarrow$ "The derivative of $f(x)$ equals $2x$ "

Notations for derivatives:

$$\begin{aligned} \frac{d}{dx} \\ y &\rightarrow y' \\ y &\rightarrow \frac{dy}{dx} \\ f(x) &\rightarrow f'(x) \end{aligned}$$

Let's try another example.

Example 2: What is $\frac{d}{dx} (10x)$?

We know $f(x) = 10x$, and can calculate $f(x+\Delta x)$:

Start with: $f(x+\Delta x) = 10(x+\Delta x)$

Expand $10(x + \Delta x)$: $f(x+\Delta x) = 10x + 10(\Delta x)$

The slope formula:

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Put in $f(x+\Delta x)$ and $f(x)$:

$$\frac{10x + 10(\Delta x) - 10x}{\Delta x}$$

Simplify ($10x$ and $-10x$ cancel):

$$\frac{10 \Delta x}{\Delta x}$$

Simplify more (divide through by Δx): $= 10$

And then as Δx heads towards 0 we get:

$$\frac{d}{dx} (10x) = 10$$

Power Rule for Differentiation

POWER RULE

$$\frac{d}{dx} ax^n = an x^{n-1}$$

Common Functions	Function	Derivative
Constant	C	0
variable	X	1
Square	x^2	$2x$
Square Root	$\sqrt{x}$	$(\frac{1}{2})x^{-\frac{1}{2}}$

Rules	Function	Derivative
Multiplication by constant	Cf	cf'
Power Rule	x^n	nx^{n-1}
Sum Rule	$f + g$	$f' + g'$
Difference Rule	$f - g$	$f' - g'$

Example 3: What is $dy/dx(x^3)$?

The question is asking "what is the derivative of x^3 ?"

We can use the Power Rule, where $n=3$:

$$\frac{dy}{dx} (x)^n = n \cdot x^{n-1}$$

$$\frac{dy}{dx} (x)^3 = 3 \cdot x^{3-1} = 3x^2$$

Example 4: What is $dy/dx \quad 5x^3$?

The derivative of $cf = cf'$

The derivative of $5f = 5f'$

We know (from the Power Rule):

$$\frac{d}{dx} (x^3) = 3x^{3-1} = 3x^2$$

So:

$$\frac{d}{dx} 5x^3 = 5 \cdot \frac{d}{dx} x^3 = 5 \times 3x^2 = 15x^2$$

Example 5: What is the derivative of $x^2 + x^3$?

The Sum Rule says:

the derivative of $f + g = f' + g'$

So we can work out each derivative separately and then add them.

Using the Power Rule:

- $\frac{d}{dx} (x^2) = 2x$
- $\frac{d}{dx} x^3 = 3x^2$

And so:

The derivative of $x^2 + x^3 = 2x + 3x^2$

Example: What is $(5z^2 + z^3 - 7z^4)$?

Using the Power Rule:

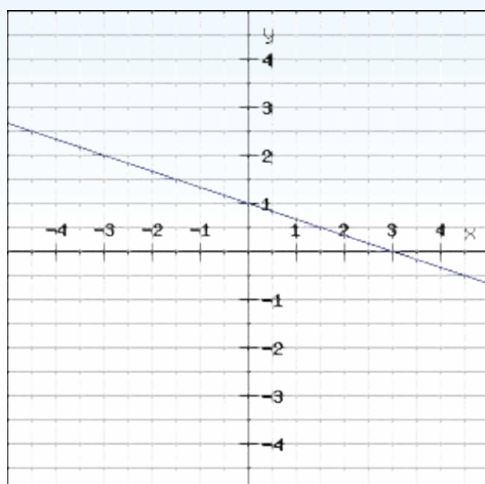
- $\frac{d}{dz} z^2 = 2z$
- $\frac{d}{dz} z^3 = 3z^2$
- $\frac{d}{dz} z^4 = 4z^3$

And so:

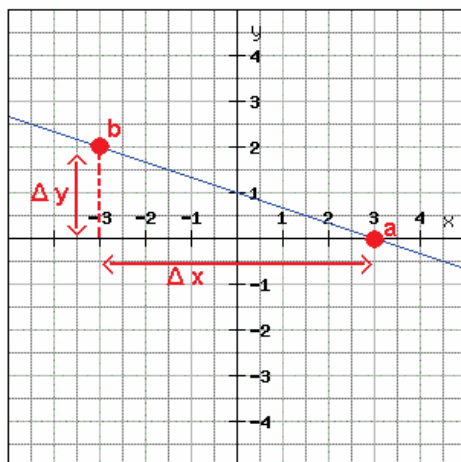
$$\frac{d}{dz} (5z^2 + z^3 - 7z^4) = 5 \times 2z + 3z^2 - 7 \times 4z^3 = 10z + 3z^2 - 28z^3$$

1. Find the slope of a straight line graphically and, knowing the equation

$y = (-1/3)x + 1$, compare the value of the derivative y' to the slope.



- a) Slope of the straight line calculation



Slope of the line $\frac{\Delta y}{\Delta x} = \frac{y_b - y_a}{x_b - x_a} = \frac{2 - 0}{-3 - 3} = -\frac{1}{3}$

- b) Derivative y' of the straight line

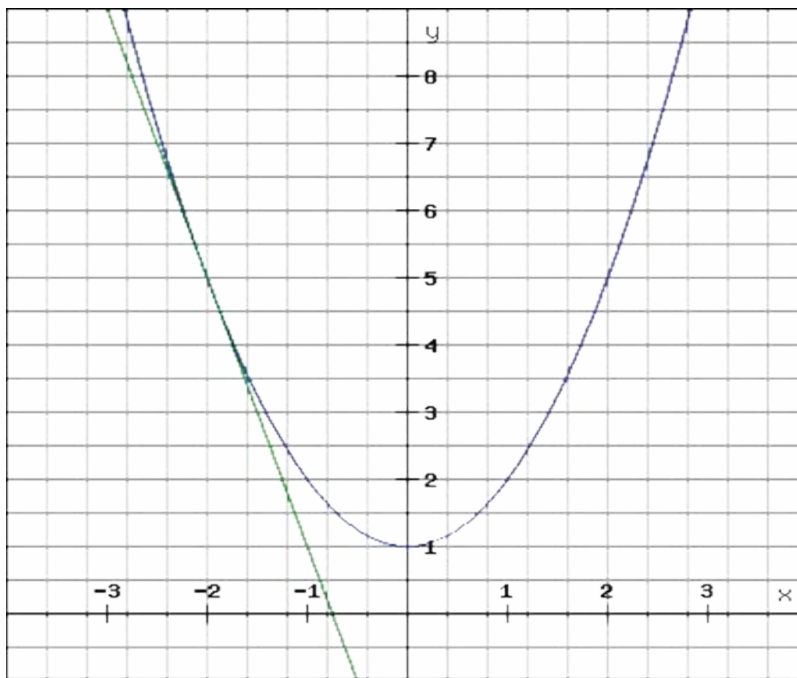
$$y' = \left(-\frac{1}{3}x + 1\right)' = -\frac{1}{3}$$

c) Reach a conclusion on the result obtained in a) and b)

The slope of the line, computed in a), is equal to the derivative of the straight line equation, computed in b). Indeed, the derivative of a linear function (general equation of a linear function) equals the slope of the line described by that function. In other words, the derivative of a linear function is the coefficient a of the function.

2. Compute the slope of a curve and its derivative :

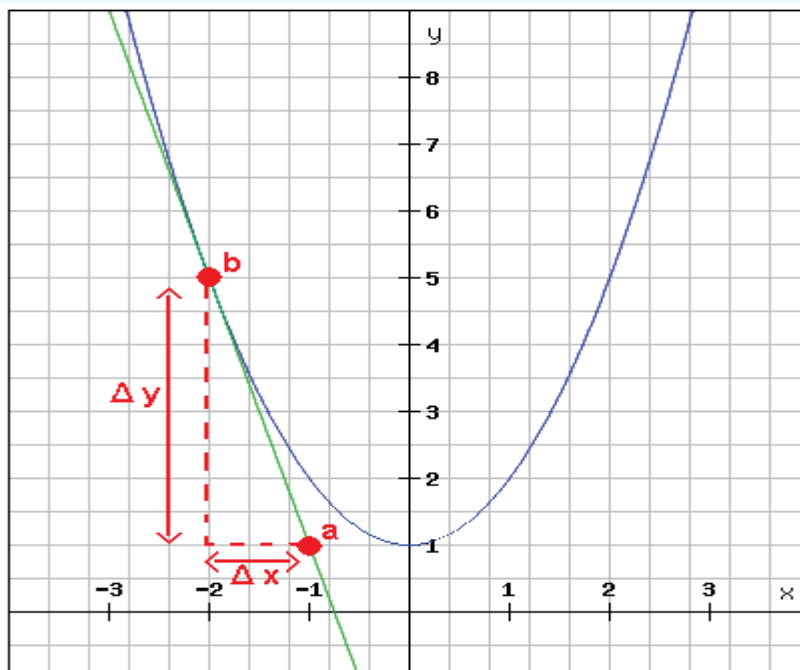
Given the graph and the quadratic equation of the following parabola $y = x^2 + 1$ and its tangent line at $x = -2$,



a) Calculation of the derivative y' of the quadratic equation (parabola equation)

$$y' = (x^2 + 1)' = 2x$$

b) Calculate graphically the slope of the tangent line to the parabola at $x = -2$



c) Find the equation of the tangent line

The general equation of a straight line is $y = ax + b$, where the coefficient a is the slope of the line.

Given that the slope of our tangent line is -4 , the equation of the tangent line will be $y = -4x + b$.

What is the value of the b term – the y -intercept?

Knowing that the tangent line is the slope of the parabola at $x = -2$, $y = (-2)^2 + 1 = 5$, giving the point $(-2, 5)$. We can now figure out the b term (y -intercept) of the line that has a slope $a = -4$ and that passes through the point $(-2, 5)$:

Tangent line equation : $y = -4x + b$, we know that $(x, y) = (-2, 5)$ satisfies this equation.

Thus $5 = -4(-2) + b$, and $5 + 4(-2) = b$ which eventually yields $b = 5 - 8 = -3$.

Conclusion : the tangent line equation is $y = -4x - 3$

d) Calculate the derivative of the tangent

$$y' = (-4x - 3)' = -4$$

e) Reach a conclusion on the results obtained in b) and d)

The derivative of the tangent line equation, computed in d), is equal to the slope of that tangent line, computed in b).

It is also interesting to notice that the derivative of the parabola is $f'(x) = (x^2 + 1)' = 2x$.

$f'(x) = 2x$ is the equation of the derivative of the parabola but it is also the equation of the slopes of the tangent lines to the parabola at various points on the curve.

In other words, $f'(x) = 2x$ expresses the slope of the curve for various values of x .

For instance, what is the value of the slope of the parabola at $x = -2$?

$$f'(-2) = 2(-2) = -4.$$

Or, for instance, what is the slope of the curve at $x = 1$?

$$f'(1) = 2(1) = 2.$$

Thus,

$\frac{dy}{dx}$ means “differentiating function ‘ y ’ with respect to the variable ‘ x ’.”

Hence,

$r = t^2 + t - 1$ has the derivative $\frac{dr}{dt} = 2t + 1$.

EXERCISE

9.1

1. Find $f'(x)$ for $f(x) = 2x^4 - 3x^3 - 7x + 2$

2. Find $f'(x)$ for $f(x) = x^{-7}$

3. Find $f'(x)$ for $f(x) = (x + 1)(x - 4)$

[Hint: Expand first]

4. Differentiate y with respect to x : $y = (x^2 - 4)(x + 2)$

5. Calculate the derivative of $f(x) = x^3$

6. What is the gradient of the tangent to the curve $y = 4x^3 - 5x^2 + x - 7$ at the point $(2, 7)$.

7. What is the derivative with respect to x of $(2x - 1)(x - 5)$?

8. Find $f'(x)$ of $f(x) = x^2 - 6x$

9. Find the derivative of $y = 7x^4$

10. Find the derivative of $y = 2x^3 - 4x^2 + 3x - 5$

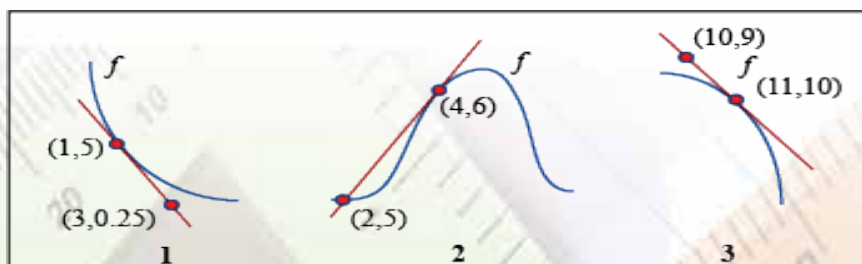
11. For the given function f and value a , find the gradient of the tangent line to f at a .

$f(x) = x^2 + 1, a = 0$

12. For the given function f and value a , find the equation of the tangent line to f at a .

- a. $f(x) = x^3, a = 1$
- b. $f(x) = 1 - x^2, a = -1$
- c. $f(x) = 2x + 3x^2, a = 4$
- d. $f(x) = 2x^3, a = -2$

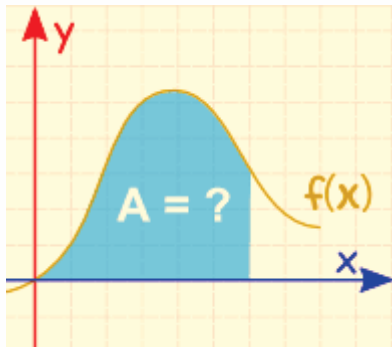
13. The graph shows a function f and a line that is tangent to f at a . For the graph determine a , $f(a)$, and $f'(a)$ (a refers to the x -value at which the line is tangent to f).



Introduction to Integration (Antidifferentiation)

Integration is the reverse of differentiation. It is a way of adding slices to find the whole.

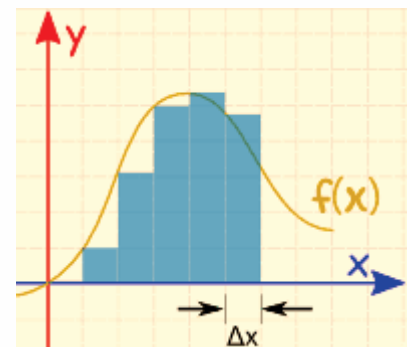
Integration can be used to find areas, volumes, central points and many useful things. But it is easiest to start with finding the **area under the curve of a function** like this:



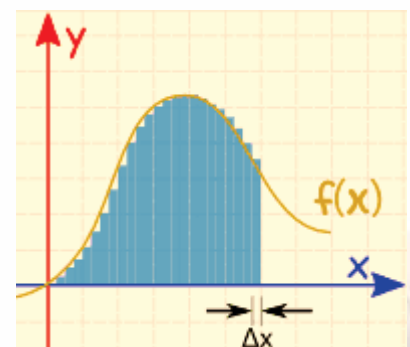
What is the area under $y = f(x)$?

Slices

We could calculate the function at a few points and **add up slices of width Δx** like this (but the answer won't be very accurate):

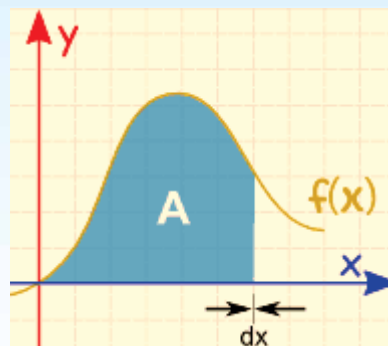


We can make Δx a lot smaller and **add up many small slices** (answer is getting better):



And as the slices **approach zero in width**, the answer approaches the **true answer**.

We now write **dx** to mean the **Δx** slices are approaching zero in width.



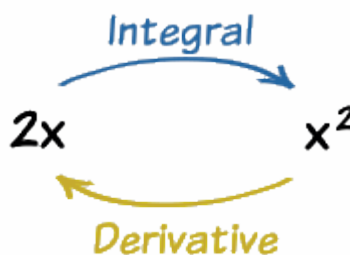
That is a lot of adding up

But we don't have to add them up, as there is a "shortcut". Remember, finding an Integral is the **reverse** of finding a Derivative.

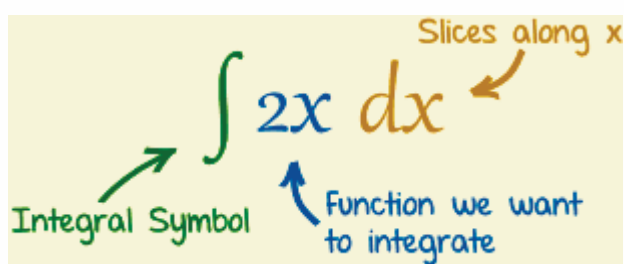
Example: What is an integral of $2x$?

We know that the derivative of x^2 is $2x$...

... so an integral of $2x$ is x^2



Notation



The symbol for "Integral" is a stylish "S" (for "Sum", the idea of summing slices):

After the Integral Symbol we put the function we want to find the integral of (called the **Integrand**), and then finish with **dx** to mean the slices go in the x direction (and approach zero in width).

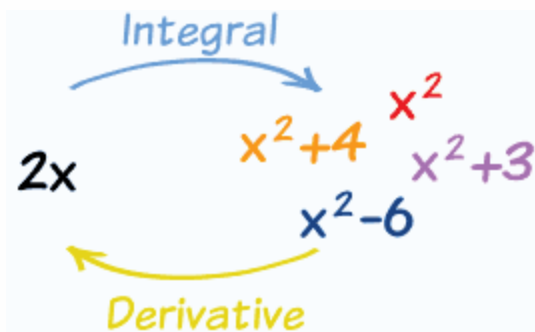
Here is how we write the answer:

$$\int 2x \, dx = \underbrace{x^2 + C}_{\text{Integral}}$$

Plus C

We wrote the answer as x^2 but why $+ C$?

It is the "Constant of Integration". It is there because of **all the functions whose derivative is $2x$** :



The derivative of $x^2 + 5$ is $2x$, and the derivative of $x^2 + 100$ is also $2x$, and so on [The derivative of a constant is zero.]

So when we **reverse** the operation (to find the integral) we only know $2x$, but there could have been a constant of any value.

Writing ' $+ C$ ' at the end represents the *constants*.

Tap and Tank

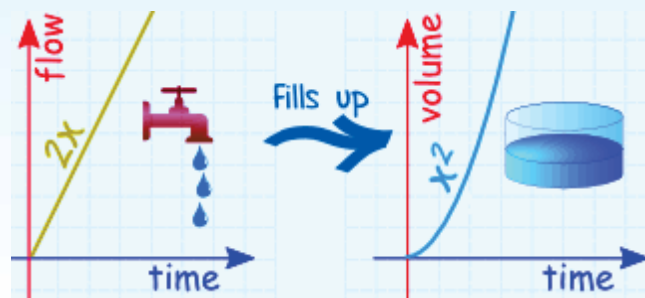


Integration is like filling a tank from a tap.

The input (before integration) is the **flow rate** from the tap.

Integrating the flow (adding up all the little bits of water) gives us the **volume of water** in the tank.

Imagine the flow starts at 0 and gradually increases (maybe a motor is slowly opening the tap).



As the flow rate increases, the tank fills up faster and faster.

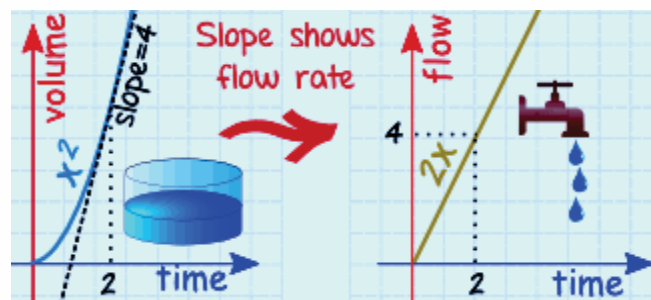
With a flow rate of $2x$, the tank fills up at x^2 .

We have integrated the flow to get the volume.

Example: (Assuming the flow is in litres per minute) after 3 minutes ($x=3$):

- the flow rate has reached $2x = 2 \times 3 = 6$ litres/min,
- and the volume has reached $x^2 = 3^2 = 9$ litres.

We can do the reverse, too:

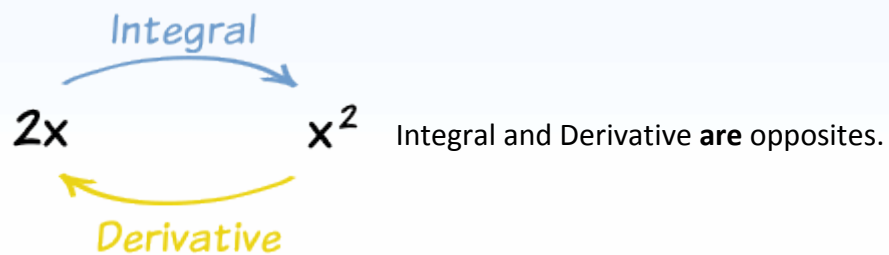


Imagine you don't know the flow rate.

You only know the volume is increasing by x^2 .

We can go in reverse (using the derivative, which gives us the slope) and find that the flow rate is $2x$.

Example: At 2 minutes the slope of the volume is 4, meaning it is increasing at 4 litres/minute, which is the flow rate. Likewise at 3 minutes the slope is 6, etc.



We can write that as:

The integral of the flow rate $2x$ tells us the volume of water: $\int 2x \, dx = x^2 + C$
 the slope of the volume increase $x^2 + C$ gives us back the flow rate: $(x^2 + C)' = 2x$



We even get a nice explanation of that "C" value ... maybe the tank already has water in it!

- The flow still increases the volume by the same amount
- And the increase in volume can give us back the flow rate.

Which teaches us to always add "+ C".

Power Rule for Integration

POWER RULE FOR INTEGRATION

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$n \neq -1$$

Integration Rules

Common Functions	Function	Integral
Constant	$\int a dx$	$ax + C$
Variable	$\int x dx$	$x^2/2 + C$
Square	$\int x^2 dx$	$x^3/3 + C$

Rules	Function	Integral
Multiplication by constant	$\int cf(x) dx$	$c \int f(x) dx$
Power Rule ($n \neq -1$)	$\int x^n dx$	$x^{n+1}/(n+1) + C$
Sum Rule	$\int (f + g) dx$	$\int f dx + \int g dx$
Difference Rule	$\int (f - g) dx$	$\int f dx - \int g dx$

Example: What is $\int x^3 dx$?

The question is asking "what is the integral of x^3 ?"

We can use the Power Rule, where $n=3$:

$$\int x^n dx = x^{n+1}/(n+1) + C$$

$$\int x^3 dx = x^4/4 + C$$

Example: What is $\int 6x^2 dx$?

We can move the 6 outside the integral: $\int 6x^2 dx = 6 \int x^2 dx$

And now use the Power Rule on x^2 :

$$= 6 x^3/3 + C$$

Simplify:

$$= 2x^3 + C$$

Example: What is $\int 8z + 4z^3 - 6z^2 dz$?

Use the Sum and Difference Rule:

$$\int 8z + 4z^3 - 6z^2 dz = \int 8z dz + \int 4z^3 dz - \int 6z^2 dz$$

Constant Multiplication:

$$= 8 \int z dz + 4 \int z^3 dz - 6 \int z^2 dz$$

Power Rule:

$$= 8z^2/2 + 4z^4/4 - 6z^3/3 + C$$

Simplify:

$$= 4z^2 + z^4 - 2z^3 + C$$

EXERCISE

9.2

2. Integrate the following expressions

[a] $3x^2$

[b] $-2x^4$

[c] $\frac{1}{2}x^5$

[d] $3x^2 - x$

[e] $-3t^3 + 2t - 1$

[f] -100

[g] $6p^4 - 3p^2 - p$

[h] $\frac{-1}{3}r^2 - \frac{2}{5}r + \frac{4}{7}$

Find:

2. $\int -2x - 3x^3 + x^2 - 1 \, dx$

3. $\int (x - 3)^2 \, dx$ [Hint: Expand first]

4. $\int x(x + 3) \, dx$

5. $\int -2x(1 - x^2 - 3x^3) \, dx$

Extension (Challenge!!)

1. Differentiate :

a. $\frac{1}{4}x^3$

b. $\sqrt{3p}$

2. Integrate:

a. $\frac{1}{4}x^3$

c. $\frac{1}{y^4}$

b. $\sqrt{3p}$

d. $\frac{-5}{\sqrt{r}}$

