

**Basic  
Math  
II**

**Algebra**

**Graphs**

**Coordinate  
Geometry**

**Trigonometry**

**Geometry**

**Statistics**

**Probability**

**Calculus**

**1**

**2**

**3**

**4**

**5**

**6**

**7**

**8**

**9**



MINISTRY OF EDUCATION



# **YEAR 12 MATHEMATICS TEXTBOOK**



PRODUCED BY

**CURRICULUM DEVELOPMENT UNIT  
SUVA, FIJI  
2016**



# MATHEMATICS

## YEAR 12



CURRICULUM DEVELOPMENT UNIT

FIJI

2016

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# PREFACE

The development of this Textbook was entirely based on the developed Year 12 Syllabus.

It has a total of **nine** strands: Basic Mathematics 2, Algebra, Graphs, Coordinate Geometry, Trigonometry, Statistics, Probability, Geometry and Calculus and these are further divided into Sub - Strands.

The contents of this book have been simplified so that it can be used by all students with different capabilities. It contains very useful materials to help students and teachers alike to prepare for the Year 12 external examination.

It is confidently believed that it will furnish Year 12 students with the necessary number and variety of exercises essential to successful instructions in mathematics.

The step – by – step instructions in the methods and examples will make it suitable for both direct one – to – one tutoring and as well as regular classroom use. Moreover, there are inclusions of external examination [Fiji School Leaving Certificate or Fiji Year 12 Certificate Examination] questions and illustrations that will help students a great deal.

All examples that have been introduced can even be attempted by an average pupil without assistance. They have been carefully graded to suit the slow learners as well, while there are some problems that are provided for advance learners.

Teachers and students are also **advised to use other resources** for enhancing of teaching and learning. This textbook is **just a guide** to accomplish the learning outcomes.

# ACKNOWLEDGEMENTS

Throughout the process in writing this textbook, a number of people have sacrificed their valuable time to assist the Ministry of Education. They are acknowledged for their active participation and without their insights, guidance and continued support; this book may not have been possible.

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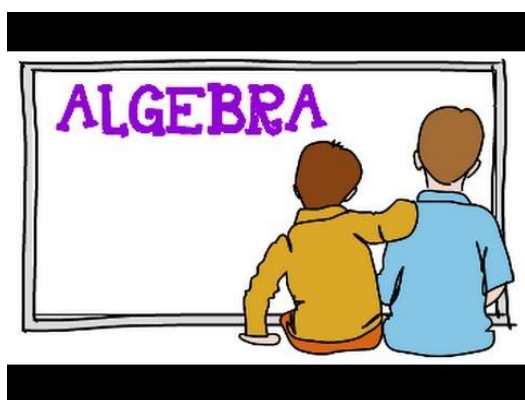
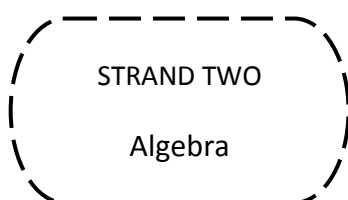
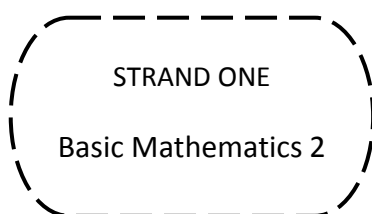
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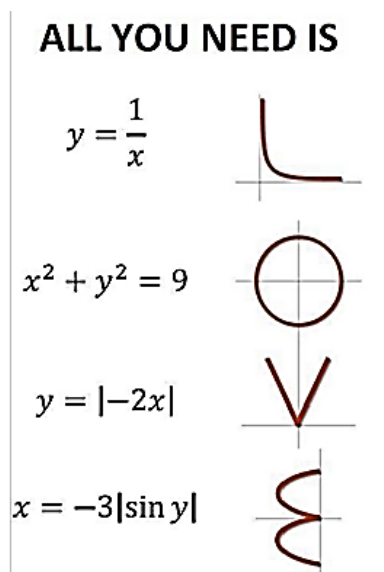
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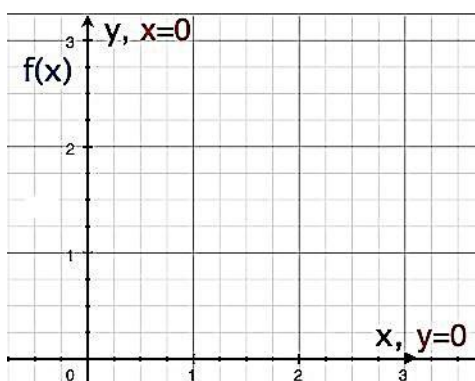
STRAND THREE

Graphs



STRAND FOUR

Coordinate Geometry



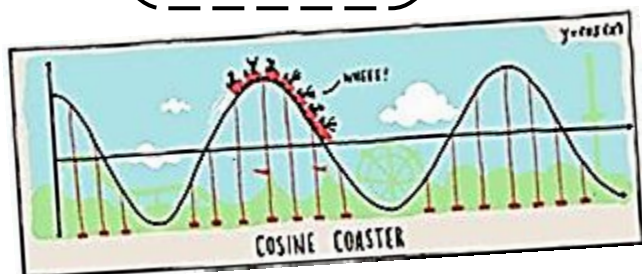
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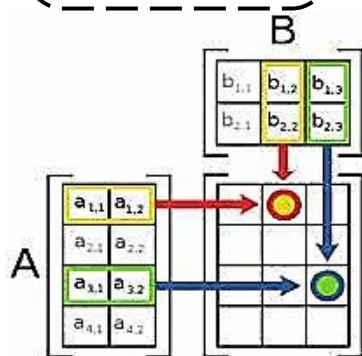
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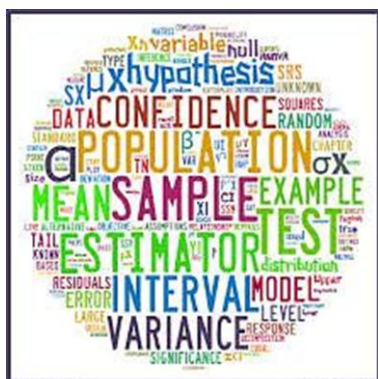


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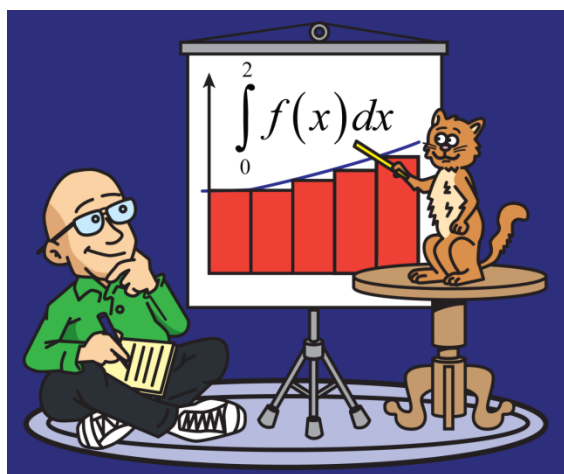
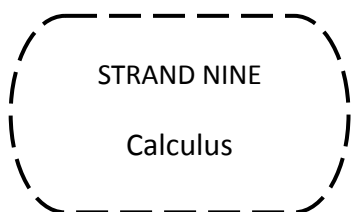
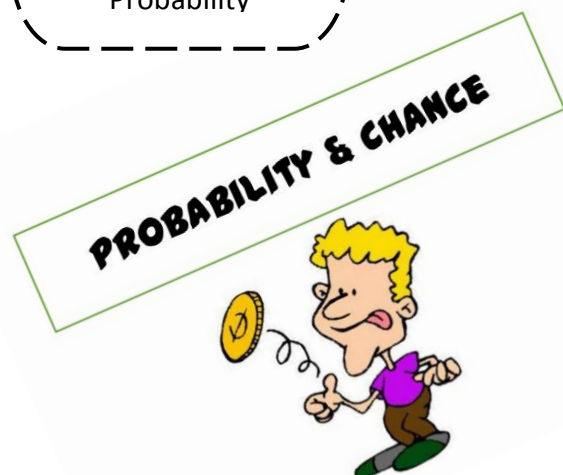
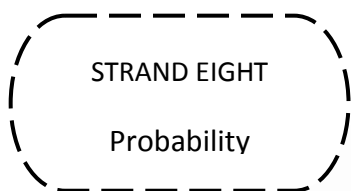
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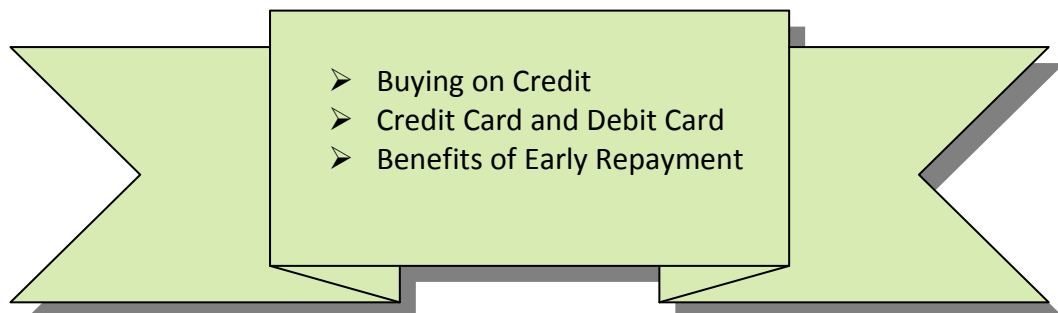
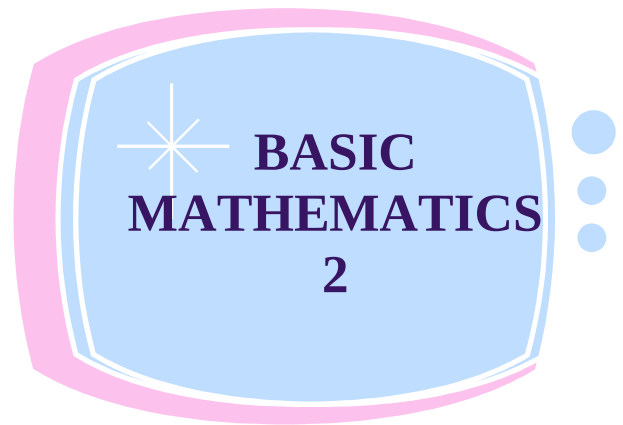
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## **BUYING ON CREDIT**

**Note:** **Credit** is borrowing money or buying goods and paying for them at a later date. It provides benefits in that it can help you make purchases by using someone else's money, which must be repaid. Credit has good and bad aspects and it needs to be managed wisely and monitored. **Interest** is the cost of credit.

### **Why should borrowing money incur interest to the borrower and earn interest for the lender?**

Interest is a charge that is paid by the borrower for the use of someone else's money. A lender expects to be rewarded for not having access to his money

There is a need to **maintain a good credit history**. It is important to establish a good credit history so that future applications for credit are viewed favorably.

On the other hand, as there is an obligation to repay money borrowed, if you borrow more than you can afford or manage to comfortably repay, credit can control you.

### **Factors that need to be considered prior to committing to regular repayments:**

- **Using savings** to make a larger down payment [for huge amounts].
- Finding a **cheaper source** of finance: research and compare and contrast with many options.
- Wait for a **sale**: if items can be bought on Cash, do so.
- **Delay purchase** until they can afford a **bigger deposit**.

**EXAMPLE 1:**

A washing machine costs \$3800. Renuka pays a deposit of \$800 and takes a loan to settle the balance over 3 years in 36 equal instalments. The interest on the loan is 12% per annum simple interest on a hire purchase loan.

Renuka is also required to insure the machine at 2% per annum of the purchase price, which is added to the loan amount.

- a) How much does Renuka pay for insuring the machine?
- b) What is the total loan amount?
- c) Calculate the total amount that Renuka paid on the instalment plan by the end of the 3 year period of the loan.
- d) What was Renuka's monthly instalment?
- e) How much did Renuka pay for the washing machine altogether?
- f) How much would Renuka have saved had she paid cash for the washing machine?

**Answers**

- a) Insurance Amount  
= 2% of \$3800  
=  $0.02 \times \$3800 \times 3 \text{ years}$   
= \$228
- b) Total Loan Amount  
= \$3800 - \$800 + \$228  
= \$3228
- c) Total amount paid on instalment plan  
= Loan + interest  
=  $3228 + 0.12 \times 3228 \times 3$   
= \$4390.08  $\approx$  \$4390.10
- d) Monthly instalment  
= (Loan + interest)  $\div$  number of instalments  
=  $\$4390.08 \div 36$   
= \$121.95
- e) Total paid  
= (Loan + interest) + Deposit  
= \$4390.08 + \$800  
= \$5190.08  $\approx$  \$5190.10
- f) Savings  
= \$5190.10 - \$3800  
= \$1390.08  $\approx$  \$1390.10

## CREDIT CARD AND DEBIT CARD

**Note:** When you use a debit card, you are using money earned yesterday to pay for something today.  
When you use a credit card, you are using money you will earn tomorrow to pay for something today.

- Debit card and Credit card both carries a bank logo and maybe a Visa/Mastercard logo.
- Debit card also known as a cheque card, works like an instant cheque. When a debit card is used, the money is taken immediately from your account to pay the supplier. It allows the holder electronic access to his/her bank account at a financial institution. An alternative to cash.

**EXAMPLE 2:** Manoj is a motor mechanic who wants to spend his annual leave in Australia. His employer will subsidize his trip up to a maximum of \$200. He has a credit card with an outstanding balance of \$350. He works out that his personal costs for this trip will be \$910.

(a) If Manoj doesn't pay off his credit card balance of \$350 and puts the cost of his trip on the card as well, how much would his credit card balance be?

(b) After returning from his trip, if he only paid the minimum amount [ 5%] off each month with interest of 1.5 % per month, how much would he still owe after four months. Show total interest cost and total of payments made.

$$\begin{aligned} \text{a) credit card balance} &= \text{Outstanding balance} + \text{trip cost} \\ &= \$350 + \$910 = \$1260 \end{aligned}$$

| (b)<br>Month | Amount<br>borrowed<br>on credit<br>card: <b>P</b> | <b>Interest</b><br>cost per<br>month at<br>1.5% per<br>month | Total amount<br>owing :<br><b>A = I + P</b> | <b>Minimum</b><br>monthly<br>repayment<br>at 5% | Total amount<br>owing carried<br>forward<br><b>Balance</b><br><b>= A - Min</b> |
|--------------|---------------------------------------------------|--------------------------------------------------------------|---------------------------------------------|-------------------------------------------------|--------------------------------------------------------------------------------|
| 1            | \$1260.00                                         | $\frac{1.5\%}{100}$ of \$1260<br>\$18.90                     | =1260.00 +<br>18.90<br>= \$1278.90          | \$63.95                                         | = \$1278.90<br>-\$63.95<br>= \$1214.95                                         |
| 2            | \$1214.95                                         | \$18.22                                                      | \$1233.17                                   | \$61.66                                         | \$1171.51                                                                      |
| 3            | \$1171.51                                         | \$17.57                                                      | \$1189.08                                   | \$59.45                                         | \$1129.63                                                                      |
| 4            | \$1129.63                                         | \$16.94                                                      | \$1146.57                                   | \$57.33                                         | \$1089.25                                                                      |

## BENEFITS OF EARLY REPAYMENT

**Note:** When you take any loan, it's **important** to understand the **structure** of your **payments**, whose components are dedicated not only to the **principal** (the amount you borrowed), but also **interest** (taxes and insurance). This structure determines how long it will take to pay off the loan.

If you make **extra payment**, it reduces the interest due on each future payment, moving towards the **ultimate goal: paying off the loan**. The first and most obvious reason for the benefits of early repayment is that it will save you tens of thousands of dollars.

**EXAMPLE 3:** Eperama has decided to take out a loan of \$20,000 to renovate his house. He looked at a number of options.

1. He finds out that he could get a personal loan for 5 years at 12% interest. The monthly payments of principal and interest will be \$444.89 per month.
2. He also considered affording payments for 4 years (monthly principal and interest payment \$526.68) or 3 years (monthly principal and interest payment \$664.29).

- (a) How much will he repay in total over the 5 years for the first loan?
- (b) How much would he repay in total if he took a four year loan?
- (c) How much would he repay in total if he took a three year loan?
- (d) Which option you would advise? Give reasons.

**Answers:**

|                        |                        |                       |
|------------------------|------------------------|-----------------------|
| (a) 5 years loan       | b) four year loan      | c) three year loan    |
| 1 yr: 12 months        | 1 yr: 12 months        | 1 yr: 12 months       |
| Thus 5 yrs = 60 months | Thus 4 yrs = 48 months | Thus 3 yrs = 36       |
| months                 |                        |                       |
| Total = 60 x \$444.89  | Total = 48 x \$526.68  | Total = 36 x \$664.29 |
| = \$26693.40           | = \$25280.64           | = \$23914.44          |

d) Three Year loan Plan: It gives the cheaper amount compared to the other 2 Plans.

Note the smaller the Year Plan, the lesser interest, thus paying less or saving more on interest.

## EXERCISE 1:

### [Multiple Choice Question]

1. A credit card will attract a high interest rate. This is because:
  - (A) It does not have a bank account with it.
  - (B) The customer can pay it off at any time.
  - (C) The credit card is usually only for small amounts of credit.
  - (D) Credit cards are unsecured lending.
  
2. Which of the following is incorrect? Credit cards are useful because
  - (A) You do not have to repay them in full each month.
  - (B) You can save on bank fees.
  - (C) The credit card gives you a period of free credit.
  - (D) They are convenient.
  
3. Under which of the following circumstances would it be financially beneficial to you to borrow money to buy something now and repay it with future income?
  - (A) When you really need a two week holiday.
  - (B) When some clothes you like go on sale.
  - (C) When the interest on the loan is greater than the interest you get on your savings.
  - (D) When you need to buy a car to get to a high paying job.
  
4. Which is TRUE about credit cards?
  - (A) You can use your credit card to receive cash in advance.
  - (B) If your credit card balance is \$1,000 and you pay \$300 by the due date, interest is charged on the unpaid balance of \$700.
  - (C) The rate of interest on your credit card is normally higher than you can earn on a transaction account at the bank.
  - (D) All of the above.
  
5. You will improve your credit worthiness by
  - (A) Having a history of paying off all debts on time in full.
  - (B) Visiting your local bank.
  - (C) Paying cash for all goods and services.
  - (D) Donating money to charity.

6. Which is FALSE about credit cards?

- (A) You will not be charged interest if you pay your closing balance off in full on the due date.
- (B) Your credit limit is a reasonable guide to how much you can spend on your credit card.
- (C) You can use your credit card overseas.
- (D) Interest is charged on a cash advance from the day of withdrawal.

**[Questions requiring working]**

7. Kaylash wants to buy a house which costs \$300,000. He decides to take a bank loan. He needs to pay a **deposit** of 10% to the bank and the remaining amount will be paid monthly for **30 years** at a fixed rate of **8% per annum**.

- a) Calculate the deposit he will have to pay.
- b) What is the remaining amount?
- c) Calculate the amount he will be paying monthly.

8. A new TV costs \$690 cash. It is available on hire purchase by paying a deposit of 15% followed by instalments of \$55.85. (15 payments)

- a) Calculate the deposit.
- b) Find the total of the instalments.
- c) Work out the total amount to be paid.
- d) What choice is better: buying on Hire purchase or buying on cash?

9. Anjana wants to buy a washing machine. The two options available to her are:

**Option 1**

Cash Sale \$2500  
\$0 Deposit  
Instalment: \$120 monthly

**Option 2**

Cash Sale \$2500  
25% Deposit  
Instalment: \$85 monthly for

- a) Calculate the total monthly instalments for two years for both the options
- b) Find the total hire purchase price to be paid.
- c) How much extra is paid for both options?
- d) Which option is better? Explain your answer.

**Lesson of Life:** *There's no reason to stereotype yourself. Doing math is like going to the gym - it's a workout for your brain and it makes you smarter-* Danica McKellar

## SUB – STRAND 1.2

## Modular System and Group



### CONTENT LEARNING OUTCOMES

- Modulo Arithmetic
- Modulo table of addition
- Modulo table of multiplication
- Conditions for a group

#### Commutative Property

You can change the order of addends.

$$4 + 7 = 7 + 4$$

#### Associative Property

You can change the grouping of addends.

$$3 + (7 + 29) = (3 + 7) + 29$$

#### Identity Property

Adding 0 to a number doesn't change a number.

$$n + 0 = n$$

#### Distributive Property

This property combines Multiplication and Addition

$$a * (b + c) = (a * b) + (a * c)$$

## MODULO ARITHMETIC

**Note:** Modulo of a number gives the **remainder** after dividing numbers.

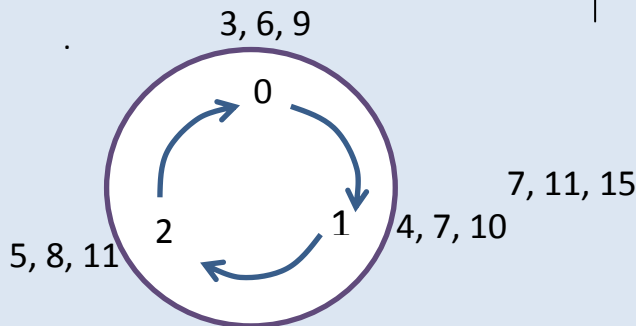
- If the first value (Numerator) is **less than** the second value (denominator), then the remainder will be same as the **first value**.
- If the first value and the second value are **same**, then the remainder will be **zero**.
- **Clock arithmetic** can be followed in order to make your work easier. The numbers can be represented in a circle. The numbers that are **less** than the modulo are **inside** the circle while the **other** numbers will be **outside** the circle. Follow the clockwise order for the rest of the numbers.

### Examples

a) Modulo 3 arithmetic

The numbers 0 – 2 will be inside the circle since it's less than 3 while 3 will be **outside** the circle at zero since 3 divided by 3 is zero.

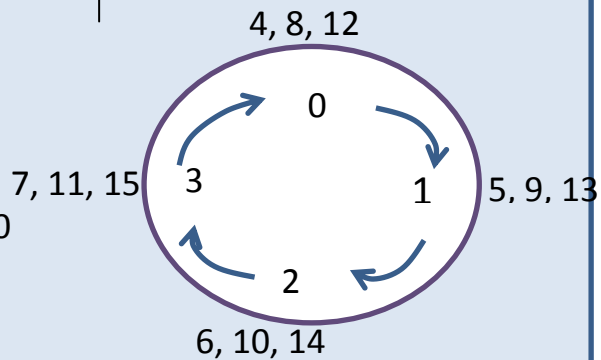
Follow the clockwise order for the rest of the numbers.



b) Modulo 4 arithmetic

The numbers 0 – 3 will be inside the circle since it's less than 4, while 4 will be at zero since remainder is zero.

Follow the clockwise order for the rest of the numbers.



- $a \bmod n$  is abbreviated as  $a \bmod n$ . From the above diagram:  
 $6 \bmod 3 = 0$   
 $10 \bmod 3 = 1$   
 $11 \bmod 3 = 2$

**Example 1:** Find the following

a)  $3 \bmod 2$

Factor  

$$\begin{array}{r} 1 \\ 2 \overline{)3} \\ \underline{-2} \\ 1 \end{array}$$
 ← Remainder  
 $\therefore 3 \bmod 2 = 1$

b)  $5 \bmod 3$

$$\begin{array}{r} 1 \\ 3 \overline{)5} \\ \underline{-3} \\ 2 \end{array}$$

$\therefore 5 \bmod 3 = 2$

c)  $4 \bmod 2$

$$\begin{array}{r} 2 \\ 2 \overline{)4} \\ \underline{-4} \\ 0 \end{array}$$

$\therefore 4 \bmod 2 = 0$

d)  $3 \bmod 4$

$$\begin{array}{r} 0 \\ 4 \overline{)3} \\ \underline{-0} \\ 3 \end{array}$$

$\therefore 3 \bmod 4 = 3$

e)  $4 \bmod 4$

$$\begin{array}{r} 1 \\ 4 \overline{)4} \\ \underline{-4} \\ 0 \end{array}$$

$\therefore 4 \bmod 4 = 0$

f)  $2 \bmod 4$

$$\begin{array}{r} 0 \\ 4 \overline{)2} \\ \underline{-0} \\ 2 \end{array}$$

$\therefore 2 \bmod 4 = 2$

## MODULO TABLE FOR ADDITION/ MULTIPLICATION

Let's see a simple addition and multiplication table up to 6 numbers.

**Example 1:** Fill the addition table for Modulo 6.

This is a normal Addition table

|                         |   |                         |   |   |   |   |    |
|-------------------------|---|-------------------------|---|---|---|---|----|
|                         |   | 2 <sup>nd</sup> element |   |   |   |   |    |
|                         | ⊕ | 0                       | 1 | 2 | 3 | 4 | 5  |
| 1 <sup>st</sup> element | 0 | 0                       | 1 | 2 | 3 | 4 | 5  |
|                         | 1 | 1                       | 2 | 3 | 4 | 5 | 6  |
|                         | 2 | 2                       | 3 | 4 | 5 | 6 | 7  |
|                         | 3 | 3                       | 4 | 5 | 6 | 7 | 8  |
|                         | 4 | 4                       | 5 | 6 | 7 | 8 | 9  |
|                         | 5 | 5                       | 6 | 7 | 8 | 9 | 10 |

The numbers that are 6 or more than 6 needs to be replaced by the **remainders**.

Thus we get,

| $\oplus$ | 0 | 1 | 2 | 3 | 4 | 5 |
|----------|---|---|---|---|---|---|
| 0        | 0 | 1 | 2 | 3 | 4 | 5 |
| 1        | 1 | 2 | 3 | 4 | 5 | 0 |
| 2        | 2 | 3 | 4 | 5 | 0 | 1 |
| 3        | 3 | 4 | 5 | 0 | 1 | 2 |
| 4        | 4 | 5 | 0 | 1 | 2 | 3 |
| 5        | 5 | 0 | 1 | 2 | 3 | 4 |

**Example 2:** Fill the multiplication table for Modulo 6.

This is a normal Multiplication table

| $\otimes$ | 0 | 1 | 2  | 3  | 4  | 5  |
|-----------|---|---|----|----|----|----|
| 0         | 0 | 0 | 0  | 0  | 0  | 0  |
| 1         | 0 | 1 | 2  | 3  | 4  | 5  |
| 2         | 0 | 2 | 4  | 6  | 8  | 10 |
| 3         | 0 | 3 | 6  | 9  | 12 | 15 |
| 4         | 0 | 4 | 8  | 12 | 16 | 20 |
| 5         | 0 | 5 | 10 | 15 | 20 | 25 |

The numbers that are 6 or more than 6 needs to be replaced by the **remainders**.

Thus we get,

| $\otimes$ | 0 | 1 | 2 | 3 | 4 | 5 |
|-----------|---|---|---|---|---|---|
| 0         | 0 | 0 | 0 | 0 | 0 | 0 |
| 1         | 0 | 1 | 2 | 3 | 4 | 5 |
| 2         | 0 | 2 | 4 | 0 | 2 | 4 |
| 3         | 0 | 3 | 0 | 3 | 0 | 3 |
| 4         | 0 | 4 | 2 | 0 | 1 | 2 |
| 5         | 0 | 5 | 4 | 3 | 2 | 1 |

**Note:** To fill the mod table, follow the same steps of modulo arithmetic, and then change those numbers in the table that are greater or equal to the mod function. The clock arithmetic is very handy here.

**Example 3:** A set  $S = \{ 0, 1, 2, 3 \}$  is given under the operation 'addition modulo 4'. Fill the table.

| $\oplus$ | 0 | 1 | 2 | 3 |
|----------|---|---|---|---|
| 0        |   |   |   |   |
| 1        |   |   |   |   |
| 2        |   |   |   |   |
| 3        |   |   |   |   |

Solution:

| $\oplus$ | 0 | 1 | 2 | 3 |
|----------|---|---|---|---|
| 0        | 0 | 1 | 2 | 3 |
| 1        | 1 | 2 | 3 | 0 |
| 2        | 2 | 3 | 0 | 1 |
| 3        | 3 | 0 | 1 | 2 |

**Exercise 2:**

1. Find the following:

a)  $11 \bmod 3$

b)  $13 \bmod 4$

c)  $10 \bmod 2$

d)  $7 \bmod 5$

2. A set  $S = \{ 0, 1, 2, 3 \}$  is given under the operation 'multiplication modulo 5'. Fill the table.

| $\otimes$ | 0 | 1 | 2 | 3 | 4 |
|-----------|---|---|---|---|---|
| 0         |   |   |   |   |   |
| 1         |   |   |   |   |   |
| 2         |   |   |   |   |   |
| 3         |   |   |   |   |   |
| 4         |   |   |   |   |   |

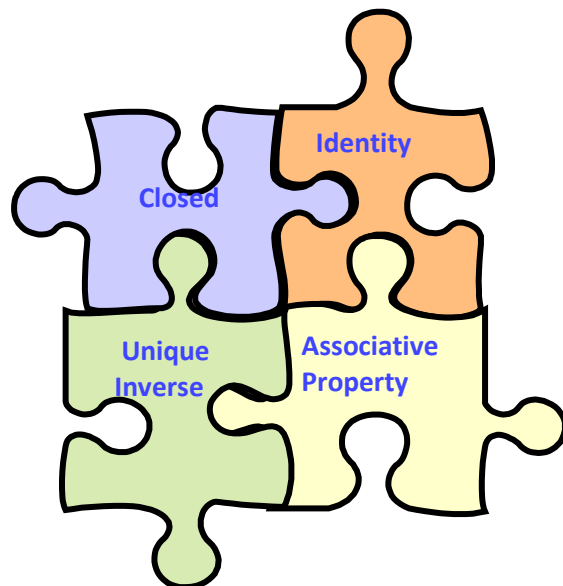
3. Part of the **addition mod 3** Table is given below. Complete the table

| $\oplus$ | 0 | 1 | 2 |
|----------|---|---|---|
| 0        | 0 |   | 2 |
| 1        | 1 |   |   |
| 2        | 2 |   |   |

## GROUP

**Note:** A set **S** is a **group** under an operation  $\star$  if the following **four** conditions are satisfied:

1.  $(S, \star)$  is **closed**.
2. There is an **identity** element for  $\star$  in  $S$ .
3. Every element in  $S$  has an **inverse** under  $\star$ .
4. The operation  $\star$  is associative.



## Definitions

1. **Closed Set or Closure of a Set** - When any pair of elements is combined, the result should not be an element outside the set (that is, no foreign elements or new elements should be seen). In other words when two elements are combined the third element is also in the set.

### 2. Identity (e):

- When an element is operated on using the identity element for that operation, the element remains unchanged:  $a * e = e * a = a$ .
- Table operation: All of the values in its row or column in the result are the same as the row or column headings. (The order of the elements in that row is equivalent to order of the elements in the operating row)

#### Shortcut Testing for Identity:

Simply highlight the row and column having the same order of elements as given in the set. The place where the two meet will be the identity element.

| © | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 2 | 3 | 0 | 1 |
| 1 | 3 | 0 | 1 | 2 |
| 2 | 0 | 1 | 2 | 3 |
| 3 | 1 | 2 | 3 | 0 |

∴ Identity = 2

### 3. Inverse ( $A^{-1}$ ):

- A Set:  $a * a^{-1} = a^{-1} * a = e$   
It simply means that any element under inverse operation will give the identity element. In other words, when an element is operated with its inverse, the result is the identity element.
- Table:  $a$  is the first element while the identity will be the third element inside the table. To find the inverse, look up for the second element.

**Shortcut Testing for Inverse (Shortcut):**

Simply draw a straight horizontal line from the first element till you reach the identity. Look on the top element for the inverse.

| © | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 2 | 3 | 0 | 1 |
| 1 | 3 | 0 | 1 | 2 |
| 2 | 0 | 1 | 2 | 3 |
| 3 | 1 | 2 | 3 | 0 |

Identity = 2 ∴ Inverse of 1 = 3

4. **Associative:** If  $S$  is a set and  $a$ ,  $b$  and  $c$  are elements of set  $S$  then it follows that  $(a*b)*c = a*(b*c)$ .

- The element order is the same, only the position of brackets changes.
- Do brackets first.

**Example 1:** A set  $S = \{0, 1, 2, 3, 4, 5\}$  is given under the operation 'addition modulo 6'.

| $\oplus$ | 0 | 1 | 2 | 3 | 4 | 5 |
|----------|---|---|---|---|---|---|
| 0        | 0 | 1 | 2 | 3 | 4 | 5 |
| 1        | 1 | 2 | 3 | 4 | 5 | 0 |
| 2        | 2 | 3 | 4 | 5 | 0 | 1 |
| 3        | 3 | 4 | 5 | 0 | 1 | 2 |
| 4        | 4 | 5 | 0 | 1 | 2 | 3 |
| 5        | 5 | 0 | 1 | 2 | 3 | 4 |

- Is it closed?
- Give the identity element of the set  $S$ .
- Find the inverse of all elements.
- Is the operation associative? Explain.
- Is this system a group?

**Answers:**

- i. Yes, it's closed since there are no foreign elements. The elements remain  $\{0, 1, 2, 3, 4, 5\}$ .
- ii.  $e = 0$
- iii. Inverses:

| $\oplus$ | 0 | 1 | 2 | 3 | 4 | 5 |
|----------|---|---|---|---|---|---|
| 0        | 0 | 1 | 2 | 3 | 4 | 5 |
| 1        | 1 | 2 | 3 | 4 | 5 | 0 |
| 2        | 2 | 3 | 4 | 5 | 0 | 1 |
| 3        | 3 | 4 | 5 | 0 | 1 | 2 |
| 4        | 4 | 5 | 0 | 1 | 2 | 3 |
| 5        | 5 | 0 | 1 | 2 | 3 | 4 |

| $\oplus$ | 0 | 1 | 2 | 3 | 4 | 5 |
|----------|---|---|---|---|---|---|
| 0        | 0 | 1 | 2 | 3 | 4 | 5 |
| 1        | 1 | 2 | 3 | 4 | 5 | 0 |
| 2        | 2 | 3 | 4 | 5 | 0 | 1 |
| 3        | 3 | 4 | 5 | 0 | 1 | 2 |
| 4        | 4 | 5 | 0 | 1 | 2 | 3 |
| 5        | 5 | 0 | 1 | 2 | 3 | 4 |

The inverses are:  $0^{-1} = 0, 1^{-1} = 5, 2^{-1} = 4, 3^{-1} = 3,$   
 $4^{-1} = 2, 5^{-1} = 1$

iv. The associative property is given as  $(a * b) * c = a * (b * c)$

Take **a** as **0**, **b** as **1** and **c** as **2**, substitute the values and use the table to show that left- hand side (LHS) is equal to right- hand side (RHS).

Do brackets first

|                     |                     |
|---------------------|---------------------|
| $LHS = (a * b) * c$ | $RHS = a * (b * c)$ |
| $LHS = (0 * 1) * 2$ | $RHS = 0 * (1 * 2)$ |
| $= 1 * 2$           | $= 0 * 3$           |
| $= 3$               | $= 3$               |

Since LHS = RHS, it is associative

iv. Since it satisfies all the four conditions, the system is a group.

**Example 2:** The table shows  $(\text{mod } 3, \oplus)$ .

| $\oplus$ | 0 | 1 | 2 |
|----------|---|---|---|
| 0        | 0 | 1 | 2 |
| 1        | 1 | 2 | 0 |
| 2        | 2 | 0 | 1 |

Is this system a group?

**Answers:**

Show the four properties:

(i) It's closed since there are no new elements

| $\oplus$ | 0 | 1 | 2 |
|----------|---|---|---|
| 0        | 0 | 1 | 2 |
| 1        | 1 | 2 | 0 |
| 2        | 2 | 0 | 1 |

(ii) There is an Identity element,  $e = 0$

(iii) Inverses are  $0^{-1} = 0$ ,  $1^{-1} = 2$ ,  $2^{-1} = 1$

(iv) It is Associative

$$\begin{aligned}
 (0 \oplus 1) \oplus 2 &= 0 \oplus (1 \oplus 2) \\
 \downarrow & \qquad \qquad \downarrow \\
 1 \oplus 2 &= 0 \oplus 0 \\
 0 &= 0
 \end{aligned}$$

| $\oplus$ | 0 | 1 | 2 |
|----------|---|---|---|
| 0        | 0 | 1 | 2 |
| 1        | 1 | 2 | 0 |
| 2        | 2 | 0 | 1 |

Since all the conditions are satisfied, it is a group.

### Exercise 3:

#### [Multiple Choice Question]

1. An operation " $\blacksquare$ " on a set  $M = \{0, 1, 2\}$  is defined by the table given below:

| $\blacksquare$ | 0 | 1 | 2 |
|----------------|---|---|---|
| 0              | 2 | 0 | 1 |
| 1              | 0 | 1 | 2 |
| 2              | 1 | 2 | 0 |

Which of the following is not **true** about the system  $(M, \blacksquare)$ ?

- A.  $M$  is closed under the operation
- B. There is no identity element.
- C.  $1 \blacksquare 1 = 1$
- D. The inverse of 2 is 0.

#### [Questions requiring working]

2. The table on operation  $\otimes$  is shown below. Is this system a group?

| $\otimes$ | 1 | 2 | 3 | 4 | 5 |
|-----------|---|---|---|---|---|
| 1         | 1 | 2 | 3 | 4 | 5 |
| 2         | 2 | 4 | 0 | 2 | 4 |
| 3         | 3 | 0 | 3 | 0 | 3 |
| 4         | 4 | 2 | 0 | 4 | 2 |
| 5         | 5 | 4 | 3 | 2 | 1 |

3. The table on operation  $\otimes$  is shown below.

| $\otimes$ | 2 | 4 | 6 | 8 |
|-----------|---|---|---|---|
| 2         | 4 | 8 | 2 | 6 |
| 4         | 8 | 6 | 4 | 2 |
| 6         | 2 | 4 | 6 | 8 |
| 8         | 6 | 2 | 8 | 4 |

- Is it closed ?
- Find the identity element.
- Find the inverses of 2, 4, 6 and 8.
- Is the operation associative? Explain.
- Is this system a group?

4. The table given below shows the result of set  $A = \{w, x, y, z\}$  under the operation  $@$ .

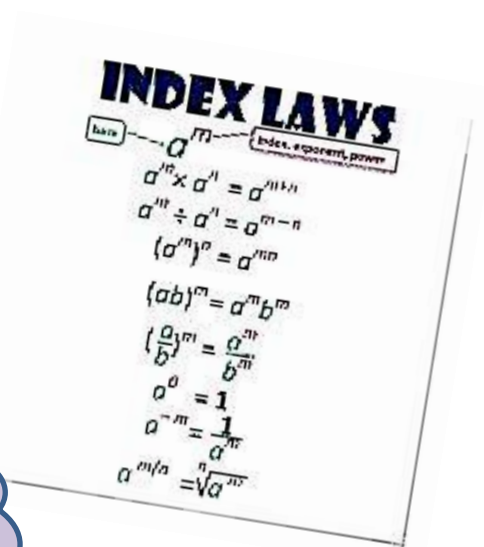
| @ | w | x | y | z |
|---|---|---|---|---|
| w | y | z | w | x |
| x | z | w | x | y |
| y | w | x | y | z |
| z | x | y | z | w |

- Find the value of  $(w @ y) @ (z @ x)$ .
- Is the operation  $@$  closed?
- Find the inverses of w and x respectively.
- Is this system a group?

**Lesson of Life:** *Mathematics proficiency is the gateway to a number of incredible careers that students may never have considered – Danica Mckellar*

## SUB – STRAND 1.3

## Indices and Logarithms

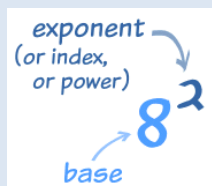


### CONTENT LEARNING OUTCOMES

- Law of indices
- Simplifying using laws of indices
- Laws of Logarithms
- Simplifying using laws of Logarithms
- Solving equations involving indices
- Surds and its conjugate
- Simplifying Surds
- Rationalizing the denominator

## LAWS OF INDICES

**Note:** It means how many times an algebraic expression is multiplied by itself.



In  $8^2$  the "2" says to use 8 twice in a multiplication, so  $8^2 = 8 \times 8 = 64$   
*Exponents are also called Powers or Indices.*

So in general:

$a^n$  tells you to multiply  $a$  by  
itself  $n$  times:

$$a^n = \underbrace{a \times a \times \dots \times a}_n$$

Some Laws are given below:

- $a^0 = 1, a \neq 0$
- $a^m \cdot a^n = a^{m+n}$
- $a^m \div a^n$  or  $\frac{a^m}{a^n}$  or  $a^m / a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^m = (a^1 b^1)^m = a^m b^m$
- $\left(\frac{a}{b}\right)^m = \left(\frac{a^1}{b^1}\right)^m = \frac{a^m}{b^m}$
- $a^{-1} = \frac{1}{a}, \quad a^{-n} = \frac{1}{a^n}$
- $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m = \frac{b^m}{a^m}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

## SIMPLIFYING USING LAWS OF INDICES

**Example 1:** Simplify  $\left(\frac{\sqrt{16x^4y^8} \cdot x^{-2}}{8y^3}\right)^{-2}$

a) square root means to raise to the power half:

$$\sqrt{16x^4y^8} = (16x^4y^8)^{1/2} = \sqrt{16} x^{4 \times 1/2} y^{8 \times 1/2} = 4 x^2 y^4$$

b) Substitute

$$\left(\frac{\sqrt{16x^4y^8} \cdot x^{-2}}{8y^3}\right)^{-2} = \left(\frac{4x^2y^4 \cdot x^{-2}}{8y^3}\right)^{-2}$$

c) Simplify the brackets

If two numbers with the same base are multiplied, then add the powers.

If two numbers with the same base are divided, then subtract the powers

$$\begin{aligned} & \left(\frac{4x^2y^4 \cdot x^{-2}}{8y^3}\right) \\ &= \frac{4x^2 \cdot x^{-2} y^4}{8y^3} \\ &= \frac{4}{8} x^{2+(-2)} y^{4-3} \\ &= \frac{1}{2} x^0 y \\ &= \frac{1}{2} y \quad \text{or} \quad \frac{y}{2} \end{aligned}$$

d) Include the power

$$\begin{aligned} & \left(\frac{y}{2}\right)^{-2} \\ &= \left(\frac{2}{y}\right)^2 \\ &= \frac{4}{y^2} \end{aligned}$$

**Example 2:** Simplify  $\frac{3^{x+2}}{3^x}$

**Note that:**  $3^x \cdot 3^2 = 3^{x+2}$

Replace the numerator and cancel to remove the same terms appearing in both numerator and denominator.

$$\begin{aligned} &\therefore \frac{3^{x+2}}{3^x \cdot 3^2} \\ &= \frac{3^x \cdot 3^2}{3^x \cdot 3^2} \\ &= \frac{3^x}{3^x} \\ &= 3^2 \\ &= 3 \times 3 \\ &= 9 \end{aligned}$$

**Example 3:** Simplify  $\frac{4^{n+1} \times 2}{2^{3n+3}}$

The smallest base is 2 so there is a need to change 4 and write it in terms of 2 as a base. From basic multiplication,  $4 = 2 \times 2 = 2^2$  so  $2^2$  is used to replace 4.

$\frac{2^{2(n+1)} \times 2^1}{2^{3n+3}}$  Now using Rule – when power is raised to another power, **multiply** both powers. In case of brackets, use distributive law to expand the brackets first.

$$\frac{2^{2(n+1)} \times 2^1}{2^{3n+3}} = \frac{2^{2n+2+1}}{2^{3n+3}}$$

Using the law of indices, separate the base and index.

$= \frac{2^{2n+3}}{2^{3n+3}} = \frac{2^{2n} 2^3}{2^{3n} 2^3}$  Cancel to remove  $2^3$  from both the numerator and denominator.

Now using the Rule – When dividing the terms with the same base, subtract the powers.

$$= 2^{2n} \div 2^{3n} = 2^{2n-3n} = 2^{-n} \therefore = \frac{1}{2^n}$$

**Example 4:** Simplify  $\frac{5 \times 625^{a+2}}{125^{a+3}}$

The smallest base is 5 so there is a need to change 625 and 125 in terms of 5 as a base. From basic multiplication,  $625 = 5 \times 5 \times 5 \times 5 = 5^4$  and  $125 = 5 \times 5 \times 5 = 5^3$  so  $5^4$  and  $5^3$  are used in place of 625 & 125 respectively.

$$\frac{5 \times (5^4)^{a+2}}{(5^3)^{a+3}}$$

Now using Rule – when power is raised to another power,

**multiply** both powers. In case of brackets, use distributive law to expand the brackets first and write base with index.

$$\frac{5 \times (5^4)^{a+2}}{(5^3)^{a+3}} = \frac{5^1 \times 5^{4a+8}}{5^{3a+9}}$$

Using the law of indices, separate the base and index.

$$\frac{5^1 \times 5^{4a+8}}{5^{3a+9}} = \frac{5^9 \times 5^{4a}}{5^9 \cdot 5^{3a}}$$

Cancel to remove  $5^9$  from both the numerator and denominator.

Now using the rule – When dividing the terms with the same bases, subtract the powers.

$$= \frac{5^{4a}}{5^{3a}} = 5^{4a-3a} = 5^{1a} = 5^a$$

#### Exercise 4:

1. Simplify the following:

a)  $x^4 \cdot x^{10}$

b)  $s^4 \cdot s^{-6}$

c)  $(x+1)^{\frac{1}{2}} \cdot (x+1)^{-\frac{1}{2}}$

2. Simplify the following:

a)  $x^{14} / x^{10}$

b)  $s^4 \div s^{-6}$

c)  $\frac{(x+1)^{\frac{1}{2}}}{(x+1)^{-\frac{1}{2}}}$

3. Simplify the following:

a)  $(x^{-2})^{-3}$

b)  $(s^4)^{\frac{1}{2}}$

c)  $\left((x+1)^{\frac{1}{2}}\right)^{-10}$

4. Simplify the following:

a)  $(x^{-2}y)^{-3}$

b)  $(4s^4t^{-6})^{\frac{1}{2}}$

c)  $(2(x+1)y)^2$

5. Simplify the following:

a)  $\left(\frac{-3x}{y}\right)^3$

b)  $\left(\frac{4s^4}{t^{-6}}\right)^{\frac{1}{2}}$

c)  $((x+1)/y)^2$

6. Simplify the following

a)  $\frac{4^{3n}}{2^{5n}}$

b)  $(4t^{-2})^{-1}$

c)  $\left(\frac{x+1}{4}\right)^{-1}$

7. Simplify a)  $(4x^2)^{\frac{1}{2}}$

b)  $3^{n-2} \times 3^n$ .

8. Simplify the following

a)  $\left(\frac{x}{y}\right)^{-3}$

b)  $\left(\frac{4}{t^{\frac{1}{2}}}\right)^{-2}$

c)  $\left(\frac{x-1}{y}\right)^{-4}$

9. Simplify the following:

a)  $(27a^{3/4})^{1/3} \times (9a^{-3/2})^{-1/2}$

b)  $\frac{15x^{-3}y^2}{(3x^{-2}y^4)^2}$

c)  $8^{4n}/16^{2n}$

d)  $\frac{2^{3n} \cdot 4^n}{8^{2n}}$

e)  $\frac{4^{n+1} \times 2^2}{2^{2n+5}}$

f)  $\frac{5^{3n} \cdot 27^{n-1}}{3^{2n-5} \cdot 125^{2n+1}}$

## INTRODUCTION TO LOGARITHMS

**Note:** In its simplest form, a logarithm answers the question:

**How many of *one number* do we multiply to get *another number*?**

**Example:** How many 2s do we multiply to get 8?

Answer:  $2 \times 2 \times 2 = 8$ , so we needed to multiply 3 of the 2s to get 8

So the logarithm is 3

**How to write it?**

We write as "the number of 2s we need to multiply to get 8 is 3" :

$$\log_2(8) = 3$$

So these two things are the same:

$$\underbrace{2 \times 2 \times 2}_3 = 8 \quad \leftrightarrow \quad \log_2(8) = 3$$

base

The number we are multiplying is called the "base", so we can say:  
"the logarithm of 8 to base 2 is 3"

Notice we are dealing with three numbers:

**The base:** the number we are multiplying (a "2" in the example above)

How many times to use it in a multiplication (3 times, which is the logarithm)

The number we want to get (an "8")

A logarithm can have any positive value as its base, but two log bases are more useful than the others. **The base-10**, or "common logarithm", log is popular for historical reasons, and is usually written as " $\log(x)$ ". For instance, pH (the measure of a substance's acidity or alkalinity), decibels (the measure of sound intensity), and the Richter scale (the measure of earthquake intensity) all involve base-10 logs. If a log has no base written, you should generally assume that the base is 10.

The other important log is the "natural logarithm", or base- $e$ , log, denoted as " $\ln(x)$ " and usually pronounced as "ell-enn-of- $x$ ", where  $e$  is an irrational and transcendental constant approximately equal to 2.718281828459. Just as the number  $e$  arises naturally in math and the sciences, so also does the natural log, which is why you need to be familiar with it. Note that  $\log_e x = \ln x$

The following are the laws:

#### LOGARITHMS

$$y = b^x \Leftrightarrow \log_b y = x$$

$$\log_b x + \log_b y = \log_b xy$$

$$\log_b x - \log_b y = \log_b \frac{x}{y}$$

$$y = \exp x \Leftrightarrow x = \ln y \quad (\ln y \equiv \log_e y)$$

$$\log_b x^n = n \log_b x$$

$$\log_b x = \frac{\log_a x}{\log_a b} \quad \text{In particular} \quad \log_e x = 2.3026 \times \log_{10} x$$

**Example 1:** Change to base - index form

i.  $\log_3 9 = 2$

$$9 = 3^2$$

ii.  $\log_4 64 = 3$

$$64 = 4^3$$

**Example 2:** Change to log form

i.  $5^2 = 25$

$b = 5$ ,  $a = 25$  &  $p = 2$  thus the log form will be  $\log_5 25 = 2$ .

ii.  $\left(\frac{1}{2}\right)^3 = \left(\frac{1}{8}\right)$

$b = \frac{1}{2}$ ,  $a = \frac{1}{8}$  &  $p = 3$  thus the log form will be  $\log_{\left(\frac{1}{2}\right)} \left(\frac{1}{8}\right) = 3$ .

**Example 3:** Simplify

i.  $\log x + 5 \log x$   
 $= 6 \log x$

ii.  $7 \log 3 - 5 \log 3$   
 $= 2 \log 3$

[When the variables/expressions are same, add or subtract the like terms]

**Example 4:** Express as a logarithm of a single number

1.  $\log 4 + \log 3$   
 $= \log(4 \times 3) = \log 12$
2.  $\log\left(\frac{1}{2}x\right) + \log 10 = \log\left(\frac{1}{2}x \times 10\right) = \log 5x$

**Example 5:** Express as a logarithm of a single number

3.  $\log 4 - \log 3$   
 $= \log\left(\frac{4}{3}\right)$
4.  $\log 10 - \log\left(\frac{1}{2}x\right) = \log\left(\frac{10}{\frac{1}{2}x}\right) = \log \frac{20}{x} = \log(20x^{-1})$

**Example 6:** Express as a logarithm of a single number

5.  $3\log 2 + \frac{1}{2}\log 4 - \log 4$   
 $\log 2^3 + \log 4^{\frac{1}{2}} - \log 4$   
 $= \log\left(\frac{2^3 \times 2}{4}\right) = \log\left(\frac{16}{4}\right) = \log 4$
6.  $\log x + \log 2x^3 - \log 2x$   
 $\log\left(\frac{x \times 2x^3}{2x}\right) = \log x^3 = 3\log x$

**Example 7:** Simplify  $\frac{2\log 9}{4\log 3}$

$$\frac{2\log 9}{4\log 3} = \frac{\log 9^2}{\log 3^4} = \frac{\log(3^2)^2}{\log 3^4} = \frac{\log 3^4}{\log 3^4} = 1$$

### Exercise 5:

1. The *base – index form* of the equation;  $\log_k m = R$  is
  - A.  $k^m = R$
  - B.  $k^R = m$
  - C.  $R^k = m$
  - D.  $m^k = R$
2. Express  $7\log 2 - \frac{1}{2}\log 16 + \log 3$  as logarithm of a single number .
3.  $2\log x + \frac{1}{2}\log 16 - \log x$  when expressed as a logarithm of a single number is
  - A.  $\log 2x^2$
  - B.  $\log 4$
  - C.  $\log \left(\frac{x}{2}\right)^2$
  - D.  $\log 4x$
4. If  $\log_a 6 = 4$  and  $\log_a 5 = 3$ , find the value of  $\log_a \frac{36}{5}$  ?
5. Change to base- index form
  - a.  $\log_{\sqrt{2}} 16 = 8$
  - b.  $\log_{\sqrt{5}} 5 = 2$
  - c.  $\log_5 625 = 4$
6. Change to log form
  - a.  $2^4 = 16$
  - b.  $3^{-2} = \frac{1}{9}$
  - c.  $36^{\frac{1}{2}} = 6$
7. If  $x = \log 6$  and  $y = \log 2$ , write  $\log 72$  in terms of  $x$  and  $y$ .
8. Write as logarithm of a single number
  - a.  $3\log 7 - \log 49$
  - b.  $2\log 5 + \log 4 - \log 20$
  - c.  $\log 2 + \frac{1}{2}\log 16 - \log 4$
9. Simplify
  - a.  $\frac{\log 9 + \log 4}{\log 12 - \log 3}$
  - b.  $\frac{\log 36}{\log 6}$

## SOLVING EQUATIONS INVOLVING INDICES

**EXAMPLE 1:** Solve the following

a)  $x^2 = 9$

Opposite is square root

$$\begin{aligned}\sqrt{x^2} &= \pm\sqrt{9} \\ x &= \pm 3 \\ x &\in \{-3, 3\}\end{aligned}$$

b)  $x^3 = 8$

Cube root

$$\begin{aligned}\sqrt[3]{x^3} &= \sqrt[3]{8} \\ x &= 2\end{aligned}$$

c)  $(x+1)^2 - 2 = 23$

with variable so remove last

$$\begin{aligned}\overbrace{(x+1)^2} - 2 + 2 &= 23 + 2 \\ (x+1)^2 &= 25 \\ \sqrt{(x+1)^2} &= \pm\sqrt{25} \\ x+1-1 &= \pm 5-1 \\ x &= -6, 4 \\ x &\in \{-6, 4\}\end{aligned}$$

**EXAMPLE 2:** Solve the following

a)  $x^{-2} = 9$

$$\begin{aligned}\frac{1}{x^2} &= 9 \\ x^2 &= \frac{1}{9} \\ \sqrt{x^2} &= \pm\sqrt{\frac{1}{9}} \\ x &= \pm\frac{1}{3} \\ x &\in \left\{-\frac{1}{3}, \frac{1}{3}\right\}\end{aligned}$$

b)  $x^{-3} = 8$

$$\begin{aligned}\frac{1}{x^3} &= 8 \\ x^3 &= \frac{1}{8} \\ \sqrt[3]{x^3} &= \sqrt[3]{\frac{1}{8}} \\ x &= \frac{1}{2}\end{aligned}$$

c)  $(x+2)^{-1} = \frac{1}{3}$

$$\begin{aligned}\frac{1}{(x+2)} &= \frac{1}{3} \\ (x+2) &= 3 \\ x+2-2 &= 3-2 \\ x &= 1\end{aligned}$$

[Apply the rule, reciprocate both sides before solving. For negative indices, use law of indices before applying the opposite operation]

**EXAMPLE 3.** Solve  $x^{\frac{3}{4}} = 27$

To solve a problem that involves indices of the form  $\sqrt[n]{x^m} = x^{\frac{m}{n}} = C$ ,  $x$  is found by making the power of  $x$  equal to 1. To do that, we raise the whole term on both sides by the reciprocal of the power of  $x$ .

**Method - reciprocal of power**

$$x^{\frac{3}{4}} = 27$$

$$\left(x^{\frac{3}{4}}\right)^{\frac{4}{3}} = (27)^{\frac{4}{3}}$$

$$x^1 = 81 \quad \left[\frac{3}{4} \times \frac{4}{3} = 1\right]$$

$$x = 81$$

Note:  $\frac{4}{3}$  should be in brackets

**Example 4:** Solve for  $x$  in  $2^{3x} = 8^{2x+1}$

To solve the equation involving the different bases, **first rewrite the bases as powers of a common base then equate the powers.**

Identify the common base (that is 2).

Write the other element as base 2,  $8 = 2 \times 2 \times 2 = 2^3$ .

Now replace the 8 with the equivalent power of 2:

$$2^{3x} = (2^3)^{2x+1}$$

$2^{3x} = 2^{6x+3}$  Same base (you may ignore the base). Equate the power and solve.

$$3x = 6x + 3$$

$$3x - 6x = 3$$

$$-3x = 3$$

$$\therefore x = -1$$

**Example 5:** Solve for  $x$ :

a)  $\log_5 x = 2$

$$x = 5^2$$

$$\therefore x = 25$$

## SOLVING BY USING LOGARITHMS

**Example 6:** Solve  $\log_2 16 = x$

$$\log_2 16 = x$$
$$2^x = 16$$

$$\log 2^x = \log 16$$
$$x \log 2 = \log 16 \quad (\text{Taking log on both sides of the equation})$$

$$x = \frac{\log 16}{\log 2}$$

$$x = 4$$

**Example 7:** Solve  $2^x = 30$

$$\log 2^x = \log 30 \quad (\text{taking log on both sides of the equation})$$

$$x \log 2 = \log 30 \quad (\text{Using the rule: } \log m^n = n \cdot \log m)$$

$$x = \log 30 \div \log 2$$

$$x = 4.91$$

### Another Method

$$2^x = 30$$

$$\ln 2^x = \ln 30 \quad (\text{taking ln on both sides of the equation})$$

$$x \ln 2 = \ln 30 \quad (\text{Using the rule: } \log m^n = n \cdot \log m)$$

$$x = \ln 30 \div \ln 2$$

$$x = 4.91$$

**Example 8:** Solve  $10^{2x} = 52$

$$10^{2x} = 52$$

$$\log 10^{2x} = \log 52 \quad (\text{taking log or ln on both sides of the equation})$$

$$2x \log 10 = \log 52 \quad (\text{Using the rule: } \log m^n = n \cdot \log m)$$

$$2x = \log 52 \div \log 10$$

$$x = (\log 52 \div \log 10) \div 2$$

$$x = 0.86$$

## EXERCISE 6:

**Solve** the equation:

1.  $\log_2 x = 5$

2.  $\log_b 64 = 3$

3.  $4^{2x+1} = 64$

4.  $2^{p+1} = 32$

5.  $5^{3t-1} = 8$

6.  $\log_3 a = 4$

7.  $5^{x+1} = 125$

8.  $\log_2 9q = 5$

9.  $4^{2x+1} = 27$

10.  $\log_v 216 = 3$

11.  $\log_p (1/2) = -2$

12.  $\log_2 11 = x - 3$

13.  $7a^3 - 4 = 3$

14.  $y^{\frac{2}{3}} = 16$

15.  $2x^{\frac{2}{3}} = 7$

16.  $2p^{\frac{1}{2}} - 1 = 3$

## SURDS

When a number cannot be further simplified to remove a square root (or cube root and so on) then it is a **surd**.

For instance  $\sqrt{2}$  (square root of 2) can't be simplified further so it is a **surd** while  $\sqrt{4}$  (square root of 4) **can** be simplified (to 2), so it is **not a surd**!

In fact "Surd" used to be another name for "Irrational", but it is now used for a **root** that is irrational.

**Example 1:** Identify whether the following are surds or not.

|                       |                         |                       |
|-----------------------|-------------------------|-----------------------|
| 1. $\sqrt{3}$         | 2. $\sqrt{\frac{1}{4}}$ | 3. $\sqrt[3]{11}$     |
| 1.7320508 <b>Surd</b> | 0.5 <b>Not a surd</b>   | 2.2239800 <b>Surd</b> |

The surds have a decimal which goes on forever without repeating: they are actually known as Irrational Numbers.

## SIMPLIFYING SURDS

**Note:** To simplify surds, first find highest factor of the number where one is a square number. Let's begin with the smallest number and get the square of it. The idea here is that we use the square number on the right side.

$$\left. \begin{array}{l} 2^2 = 4 \\ 3^2 = 9 \\ 4^2 = 16 \\ \vdots \end{array} \right\}$$

Before you simplify, let's look at some properties associated.

- $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$   
 eg  $\sqrt{9} \times \sqrt{100} = \sqrt{9 \times 100} = \sqrt{900} = 30$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$   
 eg  $\sqrt{\frac{100}{25}} = \frac{\sqrt{100}}{\sqrt{25}} = \frac{10}{5} = 2$
- $a \times \sqrt{b} = a\sqrt{b}$   
 eg  $3 \times \sqrt{2} = 3\sqrt{2}$
- $a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$   
 eg  $3\sqrt{2} - \sqrt{2} = (3-1)\sqrt{2} = 2\sqrt{2}$

**Example 2:** Simplify each of the following:

|                                                                                                          |                                                                                                              |                                                                                                                                                         |
|----------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------|
| a) $\sqrt{8}$                                                                                            | b) $\sqrt{162}$                                                                                              | c) $\sqrt{\frac{72}{x^2}}$                                                                                                                              |
| Factors of 8 are 4 and 2                                                                                 | Factors of 162 are 81 and 2                                                                                  | Factors of 72 are 36 and 2                                                                                                                              |
| $\begin{aligned} \sqrt{8} &= \sqrt{4 \times 2} \\ &= \sqrt{4} \times \sqrt{2} = 2\sqrt{2} \end{aligned}$ | $\begin{aligned} \sqrt{162} &= \sqrt{81 \times 2} \\ &= \sqrt{81} \times \sqrt{2} = 9\sqrt{2} \end{aligned}$ | $\begin{aligned} \sqrt{\frac{72}{x^2}} &= \frac{\sqrt{72}}{\sqrt{x^2}} = \frac{\sqrt{36 \times 2}}{\sqrt{x^2}} \\ &= \frac{6\sqrt{2}}{x} \end{aligned}$ |

$$d) \sqrt{12} - \sqrt{3} + \sqrt{27}$$

Factorize and collect like terms

$$\begin{aligned} &= \sqrt{4 \times 3} - \sqrt{3} + \sqrt{9 \times 3} \\ &= \sqrt{4} \cdot \sqrt{3} - \sqrt{3} + \sqrt{9} \sqrt{3} \\ &= 2\sqrt{3} - \sqrt{3} + 3\sqrt{3} \\ &= \sqrt{3}(2 - 1 + 3) \\ &= 4\sqrt{3} \end{aligned}$$

$$e) 7\sqrt{3} + 3\sqrt{5} - 2\sqrt{3} - 6\sqrt{5}$$

$$\begin{aligned} &= 7\sqrt{3} + 3\sqrt{5} - 2\sqrt{3} - 6\sqrt{5} \\ &= 7\sqrt{3} - 2\sqrt{3} + 3\sqrt{5} - 6\sqrt{5} \\ &= \sqrt{3}(7 - 2) + \sqrt{5}(3 - 6) \\ &= 5\sqrt{3} - 3\sqrt{5} \end{aligned}$$

## EXPANSION OF SURDS

Note: Recall, expansion of **algebraic expressions**:

**Distributive law:** to expand the brackets, use arrows to multiply the tail value with the head arrow value.

$$a(b \pm c) = ab \pm ac$$

**FOIL:** Expand two binomials [two terms] using the guideline: **F**irst **O**uter **I**nnner **L**ast.

$$(a + b)(c + d) = ac + ad + bc + bd$$

**Example 3:** Expand and simplify

$$a) \sqrt{2}(\sqrt{50} - 1)$$

Distributive law

$$\begin{aligned} &\sqrt{2}(\sqrt{50} - 1) \\ &= \sqrt{2}\sqrt{50} - \sqrt{2} \\ &= \sqrt{2 \times 50} - \sqrt{2} \\ &= \sqrt{100} - \sqrt{2} \\ &= 10 - \sqrt{2} \end{aligned}$$

$$b) (6x + \sqrt{5})(6x - \sqrt{5})$$

FOIL or difference of square

$$\begin{aligned} &(a + b)(a - b) = a^2 - b^2 \\ &(6x + \sqrt{5})(6x - \sqrt{5}) \\ &= (6x)^2 - (\sqrt{5})^2 \\ &= 36x^2 - 5 \end{aligned}$$

$$c) 2(3 + \sqrt{2}) - (2 + \sqrt{2})$$

Collect like terms

$$\begin{aligned} &= 2 \times 3 + 2\sqrt{2} - 2 - \sqrt{2} \\ &= 6 + 2\sqrt{2} - 2 - \sqrt{2} \\ &= 6 - 2 + 2\sqrt{2} - \sqrt{2} \\ &= 4 + \sqrt{2} \end{aligned}$$

**EXERCISE 7:****[Multiple Choice Questions]**

1. Which of the following is **not** true?

A.  $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

B.  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

C.  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

D.  $(\sqrt{a})^2 = a$

2.  $(\sqrt{2} - 3\sqrt{5})^2$  when expanded is equivalent to

A.  $-43 - 6\sqrt{10}$

B.  $-43 - 6\sqrt{10}$

C.  $47 + 9\sqrt{10}$

D.  $47 - 6\sqrt{10}$

**[Questions requiring working]**

3. Simplify these surds:

(a)  $\sqrt{48} - \sqrt{27}$

b)  $\sqrt{28} + \sqrt{63} - \sqrt{112}$

c)  $\sqrt{32x^4}$

4. Expand and simplify:

(i)  $4\sqrt{7} \times 2\sqrt{3}$

(ii)  $3\sqrt{2}(\sqrt{2} - \sqrt{8})$

(iii)  $(\sqrt{5} - 1)^2$

(iv)  $(1 + \sqrt{2})(\sqrt{3} + 4)$

5. Simplify:

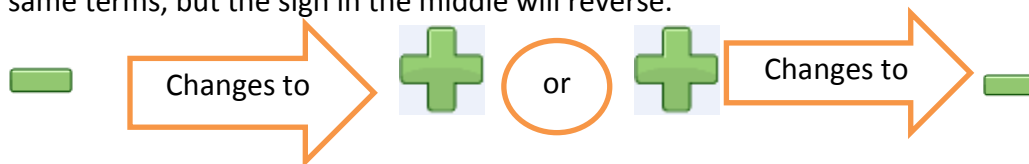
a)  $\sqrt{\frac{72}{b^2}}$

b)  $\sqrt{\frac{48}{(ab)^2}}$

c)  $\sqrt{\frac{x^2y^4}{243}}$

## CONJUGATE OF SURDS

A Binomial surd will have two terms. Its conjugate however is written with the same terms, but the sign in the middle will reverse.



**Example 1:** Find conjugate of

a)  $3 + \sqrt{2}$

Conjugate =  $3 - \sqrt{2}$

b)  $6 - \sqrt{2}$

Conjugate =  $6 + \sqrt{2}$

c)  $-\sqrt{3} - 4$

Conjugate =  $-\sqrt{3} + 4$

## RATIONALIZING THE DENOMINATOR

**Note:** It means expressing the denominator without any root sign.

- Identify the conjugate of the denominator
- Multiply the conjugate of the denominator to both the numerator and denominator
- Simplify (expand) the two brackets in numerator
- You may use 'difference of two squares' for the denominator.

**Example 2:** Simplify  $\frac{\sqrt{3}+2}{\sqrt{3}-2}$  and write in surd form

$$\begin{aligned}
 \frac{\sqrt{3}+2}{\sqrt{3}-2} \times \frac{\sqrt{3}+2}{\sqrt{3}+2} &= \frac{(\sqrt{3}+2)(\sqrt{3}+2)}{(\sqrt{3}-2)(\sqrt{3}+2)} \quad \left. \begin{array}{l} \text{Multiply the conjugate by } (\sqrt{3}+2) \text{ to} \\ \text{both the numerator and denominator} \end{array} \right\} \\
 &= \frac{\sqrt{3}\sqrt{3} + 2\sqrt{3} + 2\sqrt{3} + 4}{\sqrt{3}\sqrt{3} + 2\sqrt{3} - 2\sqrt{3} - 4} \quad \left. \begin{array}{l} \text{Expand and simplify the denominator} \\ \text{and the numerator} \end{array} \right\} \\
 &= \frac{3 + 4\sqrt{3} + 4}{3 - 4} \\
 &= \frac{7 + 4\sqrt{3}}{-1} \text{ or } -1\left(\frac{7}{1} + \frac{4}{1}\sqrt{3}\right) \text{ or } -7 - 4\sqrt{3} \quad \left. \begin{array}{l} \text{Combine and write in Surd form} \end{array} \right\}
 \end{aligned}$$

**Example 3:** Rationalize the denominator for the expression:  $\frac{3}{1-\sqrt{3}}$

$$\frac{3}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{3(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$$

$$= \frac{(3+3\sqrt{3})}{(1+\sqrt{3}-\sqrt{3}-\sqrt{3}\sqrt{3})}$$

$$= \frac{(3+3\sqrt{3})}{(1-3)}$$

$$= \frac{3+3\sqrt{3}}{-2} \text{ or } -\frac{3}{2} - \frac{3}{2}\sqrt{3}$$

**EXERCISE 8:**

Rationalize the denominator of the following:

1.  $\frac{7}{3-\sqrt{2}}$

2.  $\frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

3.  $\frac{\sqrt{2}-1}{3+\sqrt{5}}$

4.  $\frac{\sqrt{2}+1}{3-\sqrt{2}}$

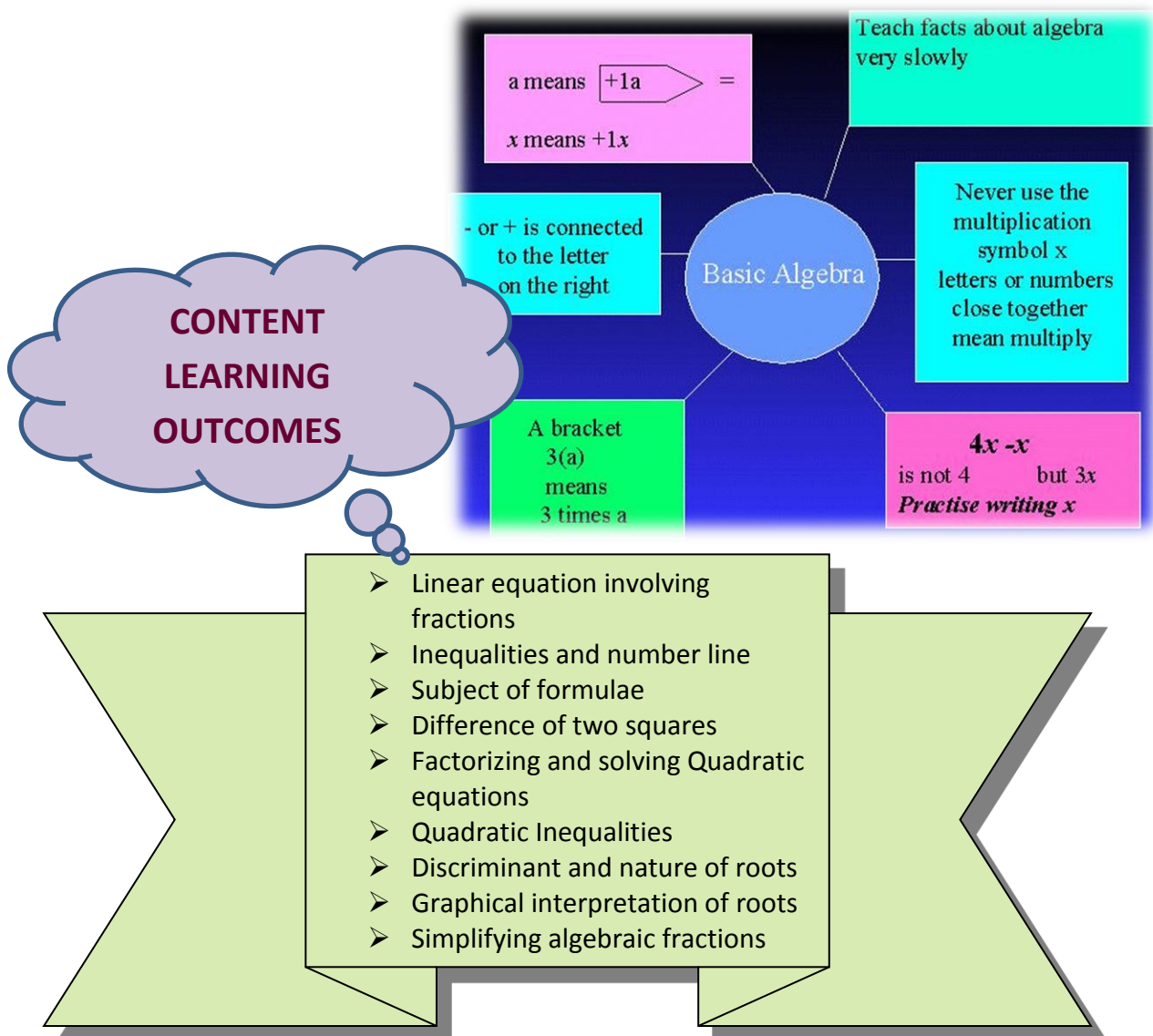
**Lesson of Life:** *Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding.* - William Paul Thurston

# STRAND TWO

# ALGEBRA

## SUB – STRAND 2.1

## Equations



## LINEAR EQUATION INVOLVING FRACTIONS

### Note:

Two important terms are **Linear** and **equation**.

**Equation** is derived from the word **equal** so it means having 'equal sign' while **Linear** is derived from the word **line**, which means that the degree/ highest power of the variable is one.

- Recall the general equation of a line is of form  $y = mx + c$ .
- To **solve equations**, follow steps of making the variable, preferably ' $x$ ' the **subject**.

If  $x$  is appearing on **both sides** of the equation, collect all  $x$  on **one side** of the equation.

The most important thing to do in **fractions** is to make **denominator the same**. You may use LCM [lowest common multiple] or cross – multiplication to get the same denominator.

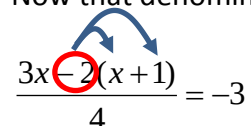
**Let's begin with solving algebraic fractions with variables on one side of the equation:**

**EXAMPLE 1: Using LCM** Solve the equation  $\frac{3x}{4} - \frac{x+1}{2} = -3$

Use LCM: the highest number in denominator is four, which is a multiple of 2. So multiply the second term by 2 [multiply to both numerator and denominator].

$$\frac{3x}{4} - \frac{x+1}{2} \times \frac{2}{2} = -3$$
$$\frac{3x}{4} - \frac{2(x+1)}{4} = -3$$

Now that denominator is same, collect the terms in the numerator together.


$$\frac{3x - 2(x+1)}{4} = -3$$

Use distributive law to expand the brackets.

$$\frac{3x - 2x - 2}{4} = -3$$

Subtract like terms

$$\frac{x-2}{4} = -3$$

Now that we have simplified the equation, solve for x.

**Note: To solve**

- ➡ Value with variable to be removed last.
- ➡ Do opposite operation on both sides of the equation.
- ➡ Continue with the above steps till the variable is written alone.

$$\begin{aligned} \frac{x-2}{4} \times 4 &= -3 \times 4 \\ x-2+2 &= -12+2 \\ \therefore x &= -10 \end{aligned}$$

**EXAMPLE 2: Using Cross – multiplication** Solve the equation  $\frac{x+1}{2} - \frac{x-2}{3} = 1$

Use cross – multiplication: Take the denominator and multiply to the opposite term. Keep in mind to write the **numerator/denominator in brackets** if it contains more than one term.

$$\frac{x+1}{2} - \frac{x-2}{3} = 1$$

$$\frac{3(x+1) - 2(x-2)}{2 \times 3} = 1$$

Also, multiply the two numbers in the denominator.

$$\frac{3(x+1) - 2(x-2)}{2 \times 3} = 1$$

Use distributive law to expand the brackets.

$$\frac{3x+3-2x+4}{6} = 1$$

Collect like terms

$$\frac{x+7}{6} = 1$$

Solve for x:

$$\begin{aligned} \frac{x+7}{6} &= 1 \\ x+7 &= 6 \Rightarrow x = -1 \end{aligned}$$

**Now let's look at solving algebraic fractions with variables appearing on both sides of the equation:**

In this case, first collect the variables on one side.

**EXAMPLE 3:** Solve for  $x$  in the equation  $\frac{x+1}{3} - 5 = \frac{-3x}{2}$

$$\frac{x+1}{3} - 5 = \frac{-3x}{2} \quad \text{Take it to the other side by adding}$$

$$\frac{x+1}{3} + \frac{3x}{2} = 5$$

$$\frac{2(x+1) + 3(3x)}{6} = 5$$

$$2(x+1) + 3(3x) = 30$$

$$2x + 2 + 9x = 30$$

$$11x + 2 = 30$$

$$11x = 28$$

$$x = 2.55$$

**If  $x$  is appearing in the denominator**, then take the reciprocal and solve.

To simplify the algebraic fractions, follow the same approach, that is get a common denominator and collect the numerator.

**EXAMPLE 4:** Solve for  $x$   $\frac{3}{x} = \frac{2}{(x-1)}$

$$\text{Reciprocate } \frac{x}{3} = \frac{(x-1)}{2}$$

Solve

$$\frac{x}{3} \times 3 = \frac{3(x-1)}{2}$$

$$x \times 2 = \frac{3(x-1)}{2} \times 2$$

$$2x = 3x - 3$$

$$2x - 2x + 3 = 3x - 2x - 3 + 3$$

$$3 = x$$

$$\therefore x = 3$$

Activity: use cross –  
multiplication to

$$\text{solve } \frac{3}{x} = \frac{2}{(x-1)}$$

### EXERCISE 9:

1. Solve for  $x$  in the equation

a)  $2x - \frac{x-3}{2} = 12$

b)  $2x - \frac{x-1}{2} = 11$

c)  $\frac{2(x-1)}{3} - (x+4) = 5$

d)  $\frac{3}{x+4} = \frac{2}{4(x-5)}$

e)  $\frac{5x+1}{3} - \frac{x}{2} = 2(x-3)$

f)  $\frac{2x-1}{2} = \frac{x-5}{5}$

g)  $x-5 = \frac{x-1}{2}$

h)  $\frac{x}{2} - 5 = \frac{x-1}{4}$

i)  $\frac{1}{x+1} = \frac{2}{x-2}$

j)  $\frac{x+1}{3} = \frac{2x}{5}$

k)  $\frac{x+1}{5} = \frac{x-3}{4}$

l)  $\frac{2}{x-5} - 3 = \frac{1}{x+2}$

**Lesson of Life:** *But mathematics is the sister, as well as the servant, of the arts and is touched with the same madness and genius* ~Harold Marston Morse

## INEQUATIONS

**Note:** Expression like  $6 > 3$ ,  $-2 < 10$  or  $3 > -9$  are called inequalities.

The sign  $>$ ,  $<$ ,  $\leq$  or  $\geq$  are used to compare numbers.

| Sign   | Meaning                  |
|--------|--------------------------|
| $>$    | Greater than             |
| $<$    | Less than                |
| $\geq$ | Greater than or equal to |
| $\leq$ | Less than or equal to    |

Solution to the inequation: inequations are solved similar to the equations.

**However, remember the rule of reversing the inequality sign when dividing or multiplying by a negative number!**

**EXAMPLE 1:** Solve

**Equation**

|                   |                  |
|-------------------|------------------|
| $5x - 1 = 3x + 3$ | Add 1            |
| $5x = 3x + 4$     | Take away $3x$   |
| $2x = 4$          | Divide by 2      |
| $x = 2$           | Is the solution. |

**Inequation**

|                   |                  |
|-------------------|------------------|
| $5x - 1 > 3x + 3$ | Add 1            |
| $5x > 3x + 4$     | Take away $3x$   |
| $2x > 4$          | Divide by 2      |
| $x > 2$           | Is the solution. |

**Example 2:** Find the solution to the inequation  $\frac{3-x}{2} \geq -1$

Solve first by making  $x$  the subject:

$$\begin{aligned} \text{L} \\ \frac{3-x}{2} &\geq -1 \\ \frac{3-x}{2} \times 2 &\geq -1 \times 2 \\ 3-x &\geq -2 \\ 3-3-x &\geq -2-3 \\ -x &\geq -5 \\ -1 &\geq -1 \quad \text{Change sign} \\ x &\leq 5 \end{aligned}$$

In Set – Builder Notation form  $\{x : x \leq 5, x \in R\}$

**Example 3:** Find the solution to the inequation  $\frac{3-x}{2} \geq \frac{x+1}{3}$

Solve first by making  $x$  the subject:

$$\begin{aligned} \frac{3-x}{2} &\geq \frac{x+1}{3} && \text{Take away the denominator by cross – multiply} \\ 3(3-x) &\geq 2(x+1) && \text{Use Distributive law} \\ 9-3x &\geq 2x+2 && \text{Collect like terms} \\ +3x &+3x \\ \text{L} \\ 9 &\geq 5x+2 && \text{Solve for } x \\ -2 &-2 \end{aligned}$$

$$\begin{aligned} \frac{7}{5} &\geq \frac{5x}{5} \\ \frac{7}{5} &\geq x \\ \therefore x &\leq \frac{7}{5} \end{aligned}$$

**EXERCISE 10:** Solve the following inequalities

1.  $\{x : -3x < 9\}$

2.  $\{x : \frac{2-x}{3} < 2\}$

3.  $\{x : \frac{2-x}{2} > -1\}$

4.  $\{x : \frac{x+1}{2} \leq \frac{x}{3}\}$

## FORMULA MANIPULATION AND SUBJECT OF THE FORMULA

### **Note:**

A **formula** is an equation which specifies how a number of variables are related to one another. Formulas are written so that a single variable is on one side of the equation. Everything else goes on the other side of the equation.

For instance,  $A = \pi r^2$ ,  $a^2 + b^2 = c^2$ ,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ,  $A = P \left(1 + \frac{r}{100}\right)^t$

For **formula manipulation**, substitute the values in place of the variable. Then use calculator to find the answer.

To **change the subject of a formula**, begin with the variable to become the new subject, and apply inverse operations as for solving equations, in the opposite order to the order convention.

**Example 1:** Make  $x$  the subject of the formula  $I = \frac{yx}{x+r}$ .

$$\begin{aligned} I &= \frac{yx}{x+r} && [\text{multiply by } (x+r)] \\ \Rightarrow I(x+r) &= yx \\ \Rightarrow Ix + Ir &= yx && [-Ix] \\ \Rightarrow Ir &= yx - Ix \\ \Rightarrow Ir &= x(y - I) && (\text{factorise}) \\ \Rightarrow \frac{Ir}{y-I} &= x. \end{aligned}$$

### [Financial Education]

**Example 2:** The formula to calculate the straight-line depreciation of an asset for a full accounting period is:

$$D = \frac{C - F}{T} \quad \text{where } D \text{ is Depreciation, } C - \text{Cost, } F - \text{Final Value and } T - \text{Time}$$

On Jan 1, 2012: Company A purchased a vehicle costing \$20,000. It is expected to have a value of \$5,000 at the end of 4 years. Calculate depreciation expense on the vehicle for the year ended Dec 31, 2015.

$$C - \$20,000, \quad F - \$5,000 \quad T - 4 \text{ years}$$
$$D = \frac{\$20,000 - \$5,000}{4} = \$3,750$$

Thus Depreciation expense is \$3,750.

**Example 3:** The formula  $C = \frac{5}{9}(F - 32)$  is used to convert temperature from Fahrenheit to degrees Celsius.

- a) If  $F = 41$ , Calculate the value of  $C$ .
- b) Make  $F$  the subject of the formula
- c) If  $C = 40^\circ$ , Calculate  $F$ .

### Answers:

- a) If  $F = 41$ , Calculate the value of  $C$   
Substitute 41 in place of  $F$

$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}(41 - 32)$$
$$= 5^\circ$$

- b) Make  $F$  the subject: Solve for  $F$

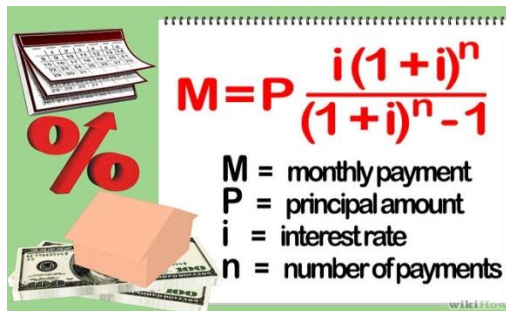
$$C = \frac{5}{9}(F - 32)$$
$$C \times \frac{9}{5} = \frac{5}{\cancel{9}} \times \frac{9}{\cancel{5}}(F - 32)$$
$$\frac{9}{5}C = F - 32$$
$$\therefore F = \frac{9}{5}C + 32$$

- c) If  $C = 40^\circ$ , Calculate  $F$ .

$$F = \frac{9}{5}C + 32$$
$$= \frac{9}{5}(40) + 32$$
$$= 122^\circ F$$

### EXERCISE 11:

1. Make  $u$  the subject of  $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ .
2. Make  $x$  the subject of  $y = \frac{2x-3}{x+4}$ .
3. Area of Circle is given as  $A = \pi r^2$ , where  $r$  is the radius of the circle.
  - a) If  $r$  is 4cm, find the area.
  - b) Make  $r$  the subject.
  - c) If  $A$  is  $12.57 \text{ cm}^2$ , find the radius.
4. An equation is given as  $F = d \times \$1.41 + r$ , where  $F$  is taxi fare,  $d$  is distance in kilometres and  $r$  is the flat charge.
  - a) Make  $d$  the subject.
  - b) If  $r$  is \$1.50 and  $f$  is \$10, Calculate  $d$ .
5. The general formula for calculating amount from compound interest is:  
 $A = P \left( 1 + \frac{r}{100} \right)^t$ , where  $P$  is the Principle Value,  $r$  is rate and  $t$  is time.
  - a) Calculate  $A$  if  $P$  is \$30 000, interest rate of 6% at the end of 4 years?
  - b) Make  $P$  the subject.
6. The formula for Mortgage Payments is shown below:



The graphic shows the mortgage payment formula  $M = P \frac{i(1+i)^n}{(1+i)^n - 1}$  in red text. To the left of the formula are illustrations of a stack of papers, a large red percentage sign, and some US dollar bills. Below the formula, the variables are defined:  $M$  = monthly payment,  $P$  = principal amount,  $i$  = interest rate, and  $n$  = number of payments. The source 'wikilife' is noted in the bottom right corner.

$$M = P \frac{i(1+i)^n}{(1+i)^n - 1}$$

$M$  = monthly payment  
 $P$  = principal amount  
 $i$  = interest rate  
 $n$  = number of payments

Calculate the monthly payment of a mortgage loan, if the principal (amount of home loan) is \$150 000, the interest rate is 6% and the payment is for 20 years.

7. An equation is given as  $i = \frac{pt}{m+pn}$ 
  - a). Make  $p$  the subject of the formula.
  - b). Find  $p$  if  $i=10$ ,  $m=2$ ,  $t=15$  and  $n= -2$ .

## **FACTORIZATION:**

### **1. DIFFERENCE OF TWO SQUARES**

Recall:  $a^2 - b^2 = (a - b)(a + b)$

The following steps may be useful:

- The minus sign should be in the middle of two square terms.
- Each term to have power of two. If square is missing, take square root and write in terms of square.

**Example 1:** Factorize  $2x^2 - 72$

$$\begin{aligned} & 2x^2 - 72 \\ &= 2(x^2 - 36) \\ &= 2(x^2 - \sqrt{36}^2) \\ &= 2(x^2 - 6^2) \\ &\therefore = 2(x - 6)(x + 6) \end{aligned}$$

**Example 2:** Factorize  $\frac{x^2}{4} - \frac{1}{y^2}$

$$\begin{aligned} & \frac{x^2}{4} - \frac{1}{y^2} \\ &= \frac{x^2}{\sqrt{4}^2} - \frac{\sqrt{1}^2}{y^2} \\ &= \frac{x^2}{2^2} - \frac{1^2}{y^2} \\ &= \left(\frac{x}{2} - \frac{1}{y}\right)\left(\frac{x}{2} + \frac{1}{y}\right) \end{aligned}$$

**Exercise 12:** Factorize the following

1.  $4x^2 - 36$

2.  $2x^2 - 32$

3.  $\frac{x^2}{16} - y^2$

4.  $3a^2 - \frac{27}{b^2}$

5.  $\frac{a^2}{25} - \frac{b^2}{4}$

6.  $9s^2 - 81t^2$

## 2. QUADRATIC EXPRESSIONS OF TYPE $(ax^2 + bx + c)$ where $a = 1$

**Note:** The general form of quadratic equation is  $ax^2 + bx + c$

One important feature is that the highest power (degree) is 2.

**Factorizing** is the process of finding the factors. So what exactly is a factor?

Factors are those numbers or variables which multiply together to get an expression.

Consider,

$$\begin{array}{c} 2 \times 3 = 6 \\ \text{Factor} \nearrow \quad \nwarrow \text{Factor} \end{array}$$

Let's begin with factorizing quadratic with  $a = 1$  in the equation, i.e.  $x^2 + bx + c$

There are three different approaches. Feel comfortable with one of the methods and you can stick to it.

### Example 1: [Basic Factorizing]

Factorize  $2y + 6$

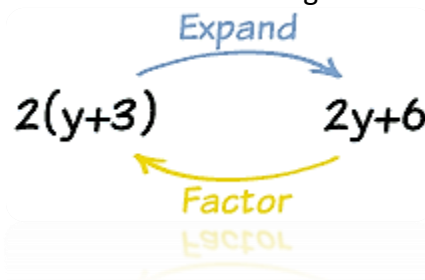
$$2y + 6$$

$$\textcircled{2}y + 3 \times \textcircled{2}$$

if you look at the two terms, you will notice that 2 is a factor of 6 so it is common in both  
change 6 and write with factors. Take out the number that is common and write the leftover numbers inside the brackets

Thus the answer is  $2(y + 3)$

Note that factoring is also the opposite of Expanding.



## FACTORISING BY GROUPING

Let's recall 'factorizing by grouping' into pairs which includes common factors.

**Example 2:** Factorize  $x^2 + 3x + 2x + 2$

$$\underbrace{3x^2 + 3x} + \underbrace{2x + 2}$$

Group in pairs. For each pair, factorize using common factors

$$\begin{array}{c} 3xx + 3x + 2x + 2 \\ \textcircled{3x}(x+1) + \textcircled{2}(x+1) \end{array}$$

You will notice that the factors in brackets will be same. This will be one of the factors. The leftover terms will be written in another bracket.

$$(x+1)(3x+2)$$

### **Factorizing Quadratics by grouping**

To factorize quadratic equation of the form  $x^2 + bx + c$ , find out two factors of the last term (c) that should add to give the middle term (b). Now replace those factors in place of  $bx$  with the variable. Use grouping technique to factorize.

**Example 3:** Factorize  $x^2 + 4x + 3$

Factors of  $3 = 3 \times 1$

Middle term  $= 3 + 1 = 4$ , let's replace

$$x^2 + \textcircled{x+3x} + 3 \quad \text{Make sure you pair in such a way that you can factorize}$$

$$\begin{array}{c} x.x + x + 3x + 3 \\ \underbrace{\phantom{x.x + x + 3x + 3}} \\ \textcircled{x}(x+1) + \textcircled{3}(x+1) \\ \phantom{x.x + x + 3x + 3} \end{array}$$
$$(x+1)(x+3)$$

### **A Short-cut Method**

#### **Method Three: Replacing factors of c**

This approach is similar to the above method.

Find out two factors of the last term (c) that should add to give the middle term (b).

Write in **brackets** with the **variable**. If numbers are **positive**, include a 'plus' sign otherwise a 'negative' sign.

**Example 4:** Factorize  $x^2 + 5x + 4$

$$\begin{array}{ccc} & b & c \\ x^2 + 5x + 4 & & 4 = 4 \times 1, \quad \text{middle term } 5 = 4 + 1 \\ & \swarrow \quad \searrow & \\ & 4 \quad \quad 1 & \end{array}$$

Thus  $(x + 4)(x + 1)$ .

**Exercise 13:** Factorize the following. Use any method

1.  $x^2 + 3x + 2$

2.  $x^2 - 3x + 2$

3.  $x^2 + 3x - 4$

4.  $x^2 - 12x + 32$

5.  $x^2 - 2x - 48$

6.  $x^2 + 2x - 15$

7.  $x^2 + 5x - 24$

8.  $x^2 - 11x + 28$

9.  $x^2 + 9x + 20$

10.  $x^2 - 10x + 25$

11.  $x^2 - 4x + 3$

12.  $x^2 + 2x - 48$

### **3. FACTORIZING QUADRATIC OF TYPE $(ax^2 + bx + c)$ where $a \neq 1$**

**Note:**

The general form of quadratic equation is  $ax^2 + bx + c$ .

Here **Grouping Technique** or the **shortcut Method** can be applied.

**Grouping Technique:** find out two factors of **ac** that should add to give the middle term (b). Now **replace** those factors in place of **bx** with the variable. Factorize in pairs.

**Shortcut Method:** Identify the **factors** of **first term** as well as the **last term**, c. **Cross – multiplication** of the terms should **add up** to give the **middle term**.

Write in **brackets**. If numbers are positive, include a 'plus' sign otherwise a 'negative' sign.

**EXAMPLE 1:** Factorize  $2x^2 - x - 6$   
Compare  $ax^2 + bx + c$   
 $ac = 2 \cdot -6 = -12$   
-4      3

Replace grouping

$$\begin{array}{rcl} 2x^2 - 4x + 3x - 6 & & \\ 2x^2 - 4x & +3x - 6 & \\ \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \\ 2x(x-2) & +3(x-2) & \\ & \nearrow & \\ (x-2)(2x+3) & & \end{array}$$

## 53

**Exercise 14:** Factorize the following. Use any method you are familiar with.

1.  $3x^2 + 11x - 20$

2.  $6c^2 + c - 12$

3.  $5p^2 - 6p + 1$

4.  $2q^2 + 7q + 6$

5.  $x^3 + 5x^2 - 6x$

6.  $2x^2 + 7x + 3$

7.  $2x^2 - 8x - 24$

8.  $2c^2 + 2c - 4$

8.  $5x^2 - 13x - 6$

10.  $3x^2 + 10x - 8$

## **SOLVING QUADRATIC EQUATIONS**

**Note:** Solving the quadratic equations, of the form  $ax^2 + bx + c = 0$ .

**Method 1: Factorise** the equation and use null factor theorem to solve for  $x$ . The Null Factor Theorem states that if  $a \cdot b = 0$ , then either  $a = 0$  or  $b = 0$ .

**Method 2: Completing the Square** Method if  $a = 1$

- Take the constant  $C$  and transpose it to the other side
- Add  $\left(\frac{b}{2}\right)^2$  to both sides. This is the square of half the coefficient of  $x$
- Write the left side as a perfect square  $\left(x + \frac{b}{2}\right)^2$  or  $\left(x - \frac{b}{2}\right)^2$
- Solve by taking the square roots

If  $a \neq 1$ , the first step would be to divide through by the coefficient. The other steps are the same.

**Method 3: Using QUADRATIC FORMULA**

This formula is very useful for solving any quadratic equations especially those which cannot be factorized using common factors. Just identify and substitute the values of  $a$ ,  $b$  and  $c$ .

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### Method 1: Factorizing and then solving

**Example 1:** Solve the equation  $x^2 + 5x + 6 = 0$

Factorize first

$$\begin{array}{ccc} & x^2 + 5x + 6 & \\ \downarrow & & \downarrow \\ x & & 2 \\ \swarrow & \searrow & \swarrow \searrow \\ & 3 & \end{array}$$

Cross – multiply  $x.3 + x.2 = 3x + 2x = 5x$  [middle term]

Solve  $(x + 2)(x + 3) = 0$

Either  $x + 2 = 0$  or  $x + 3 = 0$   
 $x = -2$   $x = -3$  Thus  $x \in \{-3, -2\}$

### Method 2: Completing the Square Method if $a = 1$

**Example 2:** Solve  $x^2 - 8x - 17 = 0$

$$x^2 - 8x = 17 \text{ (taking } c \text{ to other side)}$$
$$x^2 - 8x + 16 = 17 + 16 \text{ (add } \left(\frac{-8}{2}\right)^2 = 16 \text{ to both sides)}$$
$$(x - 4)^2 = 33 \text{ (Write the left side as a perfect square)}$$
$$x = \pm\sqrt{33} + 4$$
$$\therefore x = 9.75, x = -1.75$$

### Method 2: Completing the Square Method if $a \neq 1$

**Example 3:** Solve  $5x^2 + 8x + 1 = 0$

$$\frac{5x^2}{5} + \frac{8x}{5} + \frac{1}{5} = 0$$
$$x^2 + \frac{8}{5}x + \frac{1}{5} = 0$$
$$x^2 + \frac{8}{5}x = -\frac{1}{5}$$
$$x^2 + \frac{8}{5}x + \left(\frac{4}{5}\right)^2 = -\frac{1}{5} + \left(\frac{4}{5}\right)^2$$
$$\left(x + \frac{4}{5}\right)^2 = \frac{11}{25}$$
$$\left(x + \frac{4}{5}\right) = \pm\sqrt{\frac{11}{25}} = \pm 0.66$$
$$\left(x + \frac{4}{5}\right) = \pm 0.66$$
$$x = \pm 0.66 - \frac{4}{5}$$
$$x = -0.14, -1.46$$

**EXAMPLE 4:** Use the quadratic formula to solve  $3x^2 - 4x - 2 = 0$

Compare

$$\begin{aligned} 3x^2 - 4x - 2 &= 0 \\ ax^2 + bx + c &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3 \times -2)}}{2 \times 3} \\ &= \frac{4 \pm \sqrt{16 + 24}}{6} \\ &= \frac{4 \pm \sqrt{40}}{6} \\ &= \frac{(4 + \sqrt{40})}{6}, \frac{(4 - \sqrt{40})}{6} \\ &= 1.72, -0.39 \\ \therefore x &\in \{-0.39, 1.72\} \end{aligned}$$

**EXAMPLE 5:** Use the quadratic formula to solve  $2x^2 - 7x + 4 = 0$

Compare

$$\begin{aligned} 2x^2 - 7x + 4 &= 0 \\ ax^2 + bx + c &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2 \times 4)}}{2 \times 2} \\ &= \frac{7 \pm \sqrt{49 - 32}}{4} \\ &= \frac{7 \pm \sqrt{17}}{4} \\ &= \frac{(7 + \sqrt{17})}{4}, \frac{(7 - \sqrt{17})}{4} \\ &= 2.78, 0.72 \\ \therefore x &\in \{0.72, 2.78\} \end{aligned}$$

**Exercise 15:**

1. Using Completing the square method, solve  $ax^2 + bx + c = 0$  to show that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. Use the quadratic formula to solve the equations (2 decimal places)

a.  $x^2 + 8x + 4 = 0$

b.  $2x^2 + 6x - 1 = 0$

c.  $2x^2 - 4x - 1 = 0$

d.  $3x^2 - 4x - 2 = 0$

e.  $x^2 - 3x - 18 = 0$

f.  $1 - 2x - 3x^2 = 0$

g.  $3x - 5x^2 = 0$

h.  $0.8x^2 + 3.4x = 1$

3. Solve the equations by factorizing the expression

a.  $2x^2 - 3x + 1 = 0$

b.  $3x^2 + 2 = 5x$

c.  $4x^2 - 4x + 2 = 1$

d.  $12x^2 + 144x = 0$

4. Solve by completing the square

a.  $x^2 + 4x - 7 = 0$

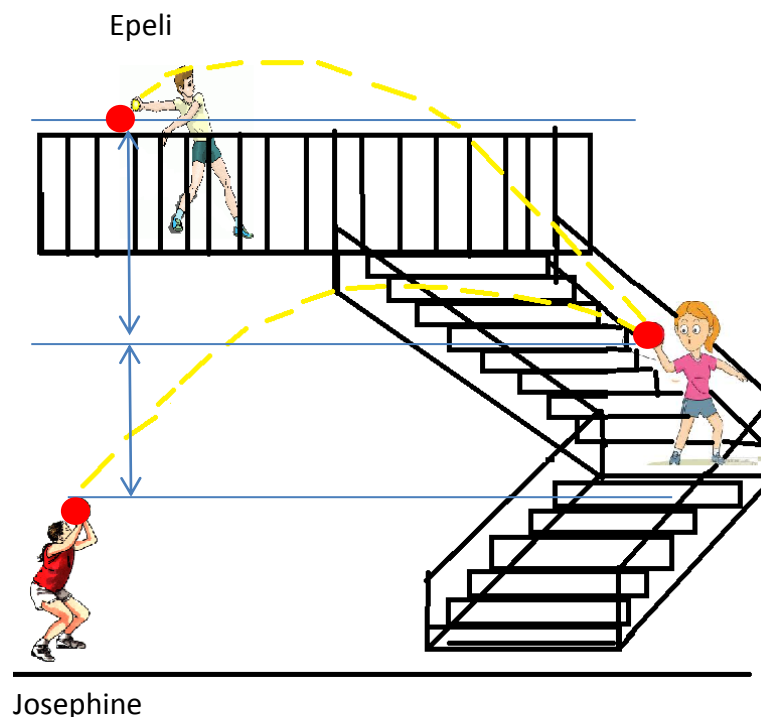
b.  $10x^2 + 3x - 2 = 0$

c.  $x^2 - 6x + 3 = 0$

d.  $2x^2 + 4x - 3 = 0$

5. A formula  $h = 3 + 14t - 5t^2$  is used to find the height [in metres from the ground] of throwing a ball into the air, where  $t$  is for time [in seconds].

- a. Josephine is throwing a ball to Divisha who is at a height of 3m, how many seconds will it take the ball to reach her?
- b. Epeli is standing at a height,  $h$ , and Divisha threw the ball to him. It took 1s to reach him. What height is Epeli from Divisha?



## QUADRATIC INEQUALITIES

**Note:** Recall solving inequalities.

For the quadratics, the following steps may be useful:

1. Solve the equation
2. Sketch the graph
3. Follow the signs
  - $<, \leq \rightarrow$  below the x-axis
  - $<, \geq \rightarrow$  above the x-axis

**Example:** Solve

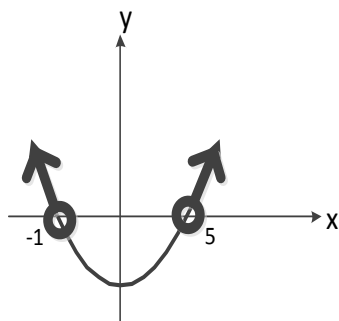
a)  $x^2 - 4x - 5 > 0$

**Answers:**

$$x^2 - 4x - 5 > 0$$

$$(x - 5)(x + 1) > 0$$

x-intercepts are 5, -1  
 $> 0$  so above the x-axis.



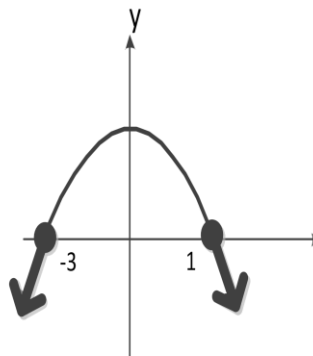
$$\therefore x < -1 \text{ or } x > 5$$

b)  $3 - 2x - x^2 \leq 0$

$$3 - 2x - x^2 \leq 0$$

$$(3 + x)(1 - x) \leq 0$$

x-intercepts are -3, 1



$$\therefore x \leq -3 \text{ or } x \geq 1$$

**Exercise 16:** Solve

1.  $3x^2 - 7x + 2 \geq 0$

2.  $2x^2 + 3x > 5$

3.  $x^2 < x$

4.  $2x^2 + 3x \leq 5$

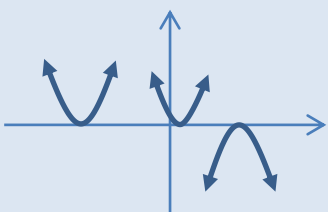
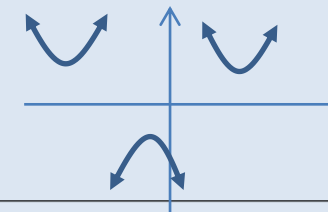
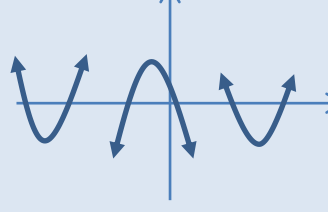
**Lesson of Life:** We use mathematics as a tool to make sense of and understand the world around us— anonymous

## DISCRIMINANT AND NATURE OF ROOTS

**Note:** The Discriminant of a quadratic equation is the expression under the square root sign of the quadratic formula, i.e.  $b^2 - 4ac$ .

The discriminant is used to determine the number of roots (solutions) of the quadratic equation.

The General Rules:

|   | Possible values of the discriminant         | Nature of roots         | Graphical interpretation of roots                                                                                    |
|---|---------------------------------------------|-------------------------|----------------------------------------------------------------------------------------------------------------------|
| 1 | discriminant is zero<br>$b^2 - 4ac = 0$     | One repeated real root  | Graph is tangential to x-axis<br> |
| 2 | discriminant is negative<br>$b^2 - 4ac < 0$ | No real roots           |                                  |
| 3 | discriminant is positive<br>$b^2 - 4ac > 0$ | Two distinct real roots |                                  |

If the discriminant is equal to zero, the polynomial is a perfect square.

E.g.  $x^2 - 6x + 9 = (x - 3)^2$  , *discriminant* = 0

**EXAMPLE 1:** Find the value of the discriminant of the equation  $\frac{x^2}{2} - x - 2 = 0$  and state its nature of roots?

$$\frac{1x^2}{2} - x - 2 = 0$$

Compare  $ax^2 + bx + c = 0$  hence  $a = \frac{1}{2}$ ,  $b = -1$  and  $c = -2$ .  
Substitute in the formula.

$$\text{Discriminant} = b^2 - 4ac = (-1)^2 - 4\left(\frac{1}{2} \times -2\right) = 1 + 4 = 5$$

Since it is positive, we expect two real roots.

**EXAMPLE 2:** A quadratic equation is given as  $-2x^2 + kx - 2 = 0$ . Find the value (s) of  $k$  such that  $-2x^2 + kx - 2 = 0$  has only 1 real root?

One real root means Discriminant = 0;  
 $a = -2$ ,  $b = k$ ,  $c = -2$

$$b^2 - 4ac = 0 \quad \text{Condition}$$

$$k^2 - 4(-2)(-2) = 0$$

$$k^2 - 16 = 0$$

$$k^2 = 16 \quad \text{Solve for } k$$

$$k^2 = 16$$

$$k = \pm\sqrt{16} = \pm 4$$

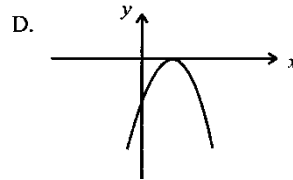
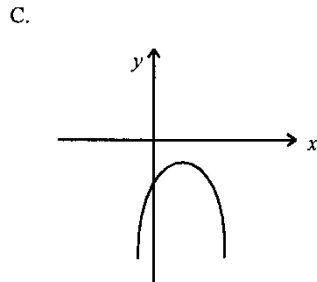
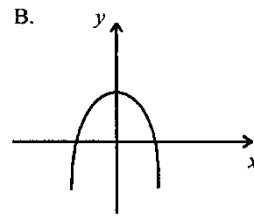
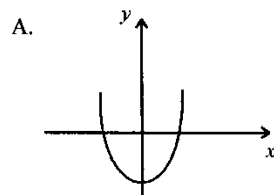
### Exercise 17:

#### [ Multiple Choice Questions]

- Given  $f(x) = 3x^2 + 3x + 1$ , the nature of the roots of  $f(x)$  is
  - one real root
  - no real root
  - two distinct real roots
  - none of the above
- The discriminant of the expression  $2x^2 - 4x - 5$  is
 

|          |         |
|----------|---------|
| A. $-10$ | B. $56$ |
| C. $10$  | D. $25$ |

3. Which of these graphs best describes a quadratic function with a negative coefficient of  $x^2$  and a negative discriminant?



**[ Fill in the blanks ]**

4. Complete the following:  
 If  $b^2 - 4ac > 0$ , the quadratic has \_\_\_\_\_ Real Roots  
 If  $b^2 - 4ac = 0$ , the quadratic has \_\_\_\_\_ Real Roots  
 If  $b^2 - 4ac < 0$ , the quadratic has \_\_\_\_\_ Real Roots

**[Questions requiring working]**

5. Find the **discriminant** and state the **nature of the roots** for the quadratic  $2x^2 + 3x + 1 = 0$
6. For what value(s) of  $p$  will the quadratic equation  $3x^2 + 4x + (p + 2) = 0$  have 2 equal roots?
7. A quadratic equation is given as  $x^2 - 6x + (p - 3) = 0$  Find the value(s) of  $p$  such that  $x^2 - 6x + (p - 3) = 0$  has 2 real roots?
8. For what values of  $p$  will the quadratic equation  $x^2 + (p + 4)x + (p + 7) = 0$  have equal roots?
9. For what value of ' $c$ ' will the expression  $y = 4x^2 - 12 + c$  have equal roots?
10. Find the value of  $k$  for which the equation  $x^2 + kx + 4 = 0$  has real roots.

**Lesson of Life:** *The essence of mathematics is not to make simple things complicated, but to make complicated things simple. (S. Gudder)*

## SIMPLIFYING ALGEBRAIC EXPRESSIONS

**Note:** In addition or subtraction, collect like terms.

- For multiplication of algebraic fractions, multiply numerators and denominators separately. Cancel if same terms are seen both in the numerator and denominator.
- In division, leave the first expression while multiply the reciprocal of the second fraction.

If many operations are involved, follow BEDMAS rule.

|          |                           |               |
|----------|---------------------------|---------------|
| <b>B</b> | <b>Brackets</b>           | <b>(    )</b> |
| <b>E</b> | <b>Exponents / powers</b> | $x^2$         |
| <b>D</b> | <b>Division</b>           | $\div$        |
| <b>M</b> | <b>Multiplication</b>     | $\times$      |
| <b>A</b> | <b>Addition</b>           | $+$           |
| <b>S</b> | <b>Subtraction</b>        | $-$           |

} whichever comes first

} whichever comes first

**EXAMPLE 1:**

Simplify  $\frac{2x}{y} - \frac{4x^2}{3y} \div \frac{8x}{9}$

Follow BEDMAS,  $\div$  comes first

$$\frac{2x}{y} - \frac{4x^2}{3y} \div \frac{8x}{9}$$

It's a fraction, let's change the sign to x and take the reciprocal of the second fraction

$$\frac{2x}{y} - \left( \frac{4x^2}{3y} \times \frac{9}{8x} \right) \quad \text{Collect the numerator and denominator separately}$$

$$= \frac{2x}{y} - \frac{36x^2}{24xy} \quad \text{Cancel and simplify the improper fraction}$$

$$= \frac{2x}{y} \times \frac{2}{2} - \frac{3x}{2y} \quad \text{Make common denominator and collect like terms}$$

$$= \frac{4x}{2y} - \frac{3x}{2y} \therefore = \frac{x}{2y}$$

**EXAMPLE 2:** Simplify:  $\frac{8p+4q}{p^2-q^2} \div \frac{6p+3q}{p-q}$

Factorize the individual algebraic expressions.

$$\frac{4(2p+q)}{(p+q)(p-q)} \div \frac{3(2p+q)}{p-q}$$

change the division sign to multiplication and take the reciprocal of the second fraction

$$\frac{\cancel{4(2p+q)}}{(p+q)\cancel{(p-q)}} \times \frac{\cancel{p-q}}{\cancel{3(2p+q)}}$$

Cancel those terms that appear both in numerator and denominator

Thus  $\frac{4}{(p+q)}$

**EXAMPLE 3:** Simplify:  $\frac{x^2-x-30}{25-x^2}$

Factorize the numerator and denominator separately:

$$x^2 - x - 30$$

$$\begin{array}{cc} x & -6 \\ x & 5 \end{array} \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}$$

$$(x-6)(x+5)$$

$$25 - x^2$$

$$\sqrt{25}^2 - x^2$$

$$5^2 - x^2$$

$$(5+x)(5-x)$$

Combine together

$$\frac{x^2-x-30}{25-x^2} = \frac{(x-6)\cancel{(x+5)}}{\cancel{(5+x)}(5-x)}$$

Thus  $\frac{x^2-x-30}{25-x^2} = \frac{(x-6)}{(5-x)}$

**Exercise 18:** Simplify the following

1.  $\frac{3x}{2} - 5y \div \frac{15y}{2x}$

2.  $\frac{a^2 - b^2}{8} \div \frac{b^2 + ab}{8a + 8}$

3.  $\frac{x - 2}{12 - 3x^2}$

4.  $\frac{x^2 - 2x - 24}{x + 4}, x \neq -4$

5.  $\frac{4x}{7} - 4y \div \frac{12y}{7x}$

6.  $\frac{2ce + df + cf + 2de}{4e^2 - f^2}$

7.  $\frac{x^2 + x - 6}{x^2 - x} \times \frac{x^2 - 1}{2x - 4}$

8.  $\frac{4x^2 + 7x - 2}{x + 2}, x \neq 2$

9.  $\frac{2x - 6}{2x^2 - 4x - 6}$

10.  $\frac{2p^2 + 7p - 15}{6p - 9}$

11.  $\frac{x^2 - x}{2x + 2} \times \frac{6x}{x^2 - 1}$

12.  $\frac{1 + \frac{1}{x}}{\frac{1}{x} - x}$

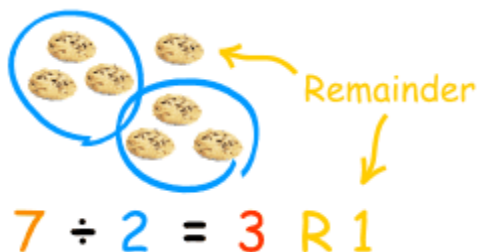
13.  $\frac{m^3 + 3m^2 + 2m}{(m+2)(m+1)}$

14.  $8cx - \frac{3c^3x^2}{2p} \div \frac{c^2x}{5p}$

**Lesson of Life:** *Many of life's failures are people who did not realize how close they were to success when they gave up.* -- Thomas Edison

## SUB – STRAND 2.2

## Remainder and Factor Theorem



"7 divided by 2 equals 3 with a **remainder of 1**"

Dividend  $\rightarrow 7 \div 2 = 3 \text{ R } 1 \leftarrow$  Remainder  
Divisor  $\uparrow$  Quotient

### CONTENT LEARNING OUTCOMES

- Long division of polynomials
- Remainder theorem
- Factor theorem

## LONG DIVISION OF POLYNOMIALS

**Note:** Recall Long Division learnt in primary schools.

When dividing one number over the other number, let's look at the case of a fraction,  $\frac{\text{Numerator}}{\text{Denominator}}$ , to change this and write in Long Division format, then

$$\begin{array}{r} \text{Factors} \\ \text{Denominator} \overline{) \text{Numerator}} \end{array}$$

A special terminology is used instead of *Numerator* and *Denominator*. Consider it below:

$$\begin{array}{r} \text{Quotient} \\ \text{Divisor} \overline{) \text{Dividend}} \end{array}$$
 The main idea here is to look for the **factors** that are closest to the **Dividend** when **Divisor is multiplied with the quotient**. The result of multiplication of **Divisor** and **quotient** will be written at the bottom of the **Dividend**. Proceed by subtracting the two numbers. Now repeat the steps till you get the remainder.

The result will be written as

$$\begin{array}{r} \text{Quotient} \\ \hline \text{Remainder} \\ \text{Divisor} \end{array}$$

For instance, **Divide 7 by 2**

$$\frac{7}{2}, \frac{\text{Numerator}}{\text{Denominator}} \Rightarrow 2 \overline{) 7}$$

Employ Long Division:

$$\begin{array}{r} 3 \\ 2 \overline{) 7} \\ - 6 \\ \hline 1 \end{array}$$

$2 \times 3 = 6$  which is closest to 7, so 3 will be the **quotient**.  
The result of **Divisor** and **quotient**.  
**Remainder**

Thus the result is  $3\frac{1}{2}$ .  
pressing  $7 \div 2$

Check using calculator by



The same method will be used to divide polynomials.

**EXAMPLE 1:** Find  $x+1 \overline{) x^3 + 7x^2 - 3x - 3}$ . What is the Remainder?

$$\begin{array}{r}
 x^2 + 6x - 9 \\
 x+1 \overline{) x^3 + 7x^2 - 3x - 3} \\
 \underline{-x^3 + x^2} \phantom{-3x - 3} \\
 6x^2 - 3x - 3 \\
 \underline{-6x^2 + 6x} \phantom{-3} \\
 -9x - 3 \\
 \underline{-(-9x - 9)} \\
 6
 \end{array}$$

Consider the first variable [left] of divisor and dividend:  $x$  times  $\_\_ = x^3$  Law of indices:  $x^2$ , so  $x^2$  will be the **quotient(on top)**.

write the result of **Divisor & quotient**, i.e.  $x^2(x+1) = x^3 + x^2$  subtract the like terms, bring the other terms down. Continue the steps:  $x$  times  $\_\_ = 6x^2$  Law of indices:  $6x$  so write  $+6x$  on the right side of the **quotient**.

The result of  $6x(x+1) = 6x^2 + 6x$  subtract

Continue the steps:  $x$  times  $\_\_ = -9x$  Law of indices:  $-9$  The result of  $-9(x+1) = -9x - 9$

6 **Remainder**

Thus the result is  $x^2 + 6x - 6 + \frac{6}{x+1}$  and Remainder = 6

**EXAMPLE 2:** Find  $x+5 \overline{) x^3 + 2x^2 - 13x + 10}$

$$\begin{array}{r}
 x^2 - 3x + 2 \\
 x+5 \overline{) x^3 + 2x^2 - 13x + 10} \\
 \underline{-x^3 + 5x^2} \phantom{-13x + 10} \\
 -3x^2 - 13x + 10 \\
 \underline{-(-3x^2 - 15x)} \phantom{+10} \\
 2x + 10 \\
 \underline{-2x + 10} \\
 0
 \end{array}$$

Consider the first variable[left] of divisor and dividend:  $x$  times  $\_\_ = x^3$  Law of indices:  $x^2$ , so  $x^2$  will be the **quotient**.

The result i.e.  $x^2(x+5) = x^3 + 5x^2$

subtract the like terms, bring the other terms down.

$x$  times  $\_\_ = -3x^2$  Law of indices:  $-3x$ , i.e.  $-3x(x+5) = -3x^2 - 15x$

subtract

$x$  times  $\_\_ = 2x$  Law of indices:  $2$ ,

The result i.e.  $2(x+5) = 2x + 10$

**Remainder**

Thus the result is  $x^2 - 3x + 2$

### Exercise 19:

1. Divide  $4x^3 - 2x^2 + 5x - 3$  by  $(x - 2)$ .
2. Divide  $x^3 - x + 3$  by  $(x + 1)$ .
3. A function is defined as  $h(x) = x^3 + 6x^2 + 5x - 12$ .

Find  $x + 4 \overline{) x^3 + 6x^2 + 5x - 12}$

4. A function is given as  $f(x) = x^3 - 6x^2 + 11x - 6$ .

Find  $x - 3 \overline{) x^3 - 6x^2 + 11x - 6}$

## REMAINDER THEOREM

**Note:** Recall Long division of polynomials. Notice that it takes a lot of time and effort to find the Remainder. Here is an easy way of finding the Remainder.

Consider a **Divisor**  $(x - a)$  which is also a factor of a polynomial,  $f(x)$ ,

- Find the value of  $x$ , i.e. by solving [**Divisor** = 0]  
 $\Rightarrow (x - a) = 0$   
 $\Rightarrow x = a$
- Substitute  $x$  value in the polynomial  $f(a)$ .
- The Remainder =  $f(a)$ .

When any polynomial  $f(x)$  is divided by  $x - a$ , the remainder is  $f(a)$ .

**EXAMPLE 1:** Calculate the remainder when  $x^3 - 2x + 1$  is divided by  $3x - 2$ .

$$\begin{array}{r} \downarrow \\ 3x - 2 = 0 \\ + 2 \quad + 2 \\ 3x = 2 \\ x = \frac{2}{3} \end{array}$$

Let Divisor = 0, Solve for  $x$

$$R = x^3 - 2x + 1$$

Substitute  $x = \frac{2}{3}$

$$\begin{aligned} &= \left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right) + 1 \\ &= -\frac{1}{27} \end{aligned}$$



**EXAMPLE 2:** Determine the value of  $a$  if  $x^3 - 3ax^2 - 8x - 8$  has a remainder of 8 when divided by  $x - 4$ .

$$x - 4 = 0$$

$$x = 4$$

$$R = x^3 - 3ax^2 - 8x - 8$$

$$8 = 4^3 - 3a(4^2) - 8(4) - 8$$

$$8 = 64 - 48a - 32 - 8$$

$$8 = 24 - 48a$$

$$-16 = -48a$$

$$\text{Thus } a = \frac{1}{3}$$

Let Divisor = 0, Solve for  $x$

Substitute  $x = 4$  and  $R = 8$

Solve for  $a$

**EXAMPLE 3:** A polynomial is given as  $g(x) = x^3 + 2x^2 + kx + m$ . When  $g(x)$  is divided by  $(x+1)$  the remainder is 3 and when  $g(x)$  is divided by  $(x - 1)$  the remainder is -1. Determine the values of  $k$  and  $m$ .

Let Divisor = 0, Solve for  $x$

$$x+1 = 0$$

$$x = -1$$

$$x - 1 = 0$$

$$x = 1$$

Substitute  $x$  values and Remainder

$$R = g(-1) = (-1)^3 + 2(-1)^2 + k(-1) + m$$

$$R = g(1) = (1)^3 + 2(1)^2 + k(1) + m$$

$$3 = g(-1) = -1 + 2 - k + m$$

$$3 = 1 - k + m$$

$$-1 - 1$$

$$2 = -k + m$$

$$-1 = g(1) = 1 + 2 + k + m$$

$$-1 = 3 + k + m$$

$$-3 - 3$$

$$-4 = k + m$$

Solve Simultaneously:

**Note:** Simultaneous Equation

It is used to solve for two variables together. Three different approaches are

- Graphical Method: Draw by finding  $x$  and  $y$  – intercepts
- Substitution Method: Substitute one equation into another equation, then solve.
- Elimination Method: One of the variables to have same coefficient, then either add/subtract to solve.

This will be discussed later in detail.

Put the two equations:

**Note:** Using Elimination Method:

- To eliminate one variable, either add or subtract the 2 equations. Remember one of the variables should get eliminated.
- Solve for one variable first
- Substitute in any equation to find the second value.

$$2 = -k + m$$

$$-4 = k + m$$

Since the coefficients are same, you can do both addition/subtraction

Add

$$2 = -k + m$$

$$-4 = k + m$$

$$\hline -2 = 2m$$

$$\text{Thus } m = -1 \text{ and}$$

Subtract

$$2 = -k + m$$

$$-4 = k + m$$

$$\hline 6 = -2k$$

$$k = -3.$$

#### Exercise 20:

##### [Multiple Choice Questions]

1.  $f(x) = 2x^3 + 2x - 3$ . When  $f(x)$  is divided by  $(x + 2)$ , the remainder is  
A. 14    B. 9    C. -23    D. -15
2. If the function  $h(x) = -x^3 - 3x^2 + bx + 5$  has a remainder of  $-2$  when divided by  $x + 2$ , what is the value of  $b$ ?  
A. -6    B. -2    C.  $\frac{3}{2}$     D. 6

##### [Questions requiring working]

3. What is the remainder when  $x^3 - 3x^2 - 4x + 7$  is divided by  $x$ ?
4. Find the remainder when  $f(x) = -2x^3 + 3x^2 - x + 5$  is divided by  $2x + 1$ .
5. Find the remainder when  $x^3 - 2x^2 + 20$  is divided by  $x - 2$ .
6. What is the remainder when  $x^2 - 3x + 5$  is divided by  $x + 2$ ?

**Lesson of Life:** *The object of education is to prepare the young to educate themselves throughout their lives* ~ Robert M. Hutchins

## FACTOR THEOREM

**Note:** Recall easy way of finding the Remainder. Only difference noted is that there will be **no** Remainder.

Consider a **Divisor**  $(x - a)$  which is also a factor of a polynomial,  $f(x)$ ,

- Find the value of  $x$ , i.e. by solving **[Divisor = 0]**  $\Rightarrow (x - a) = 0 \Rightarrow x = a$
- Substitute  $x$  value in the polynomial  $f(a)$ .
- Find the Remainder  $\boxed{f(a)}$ .

The factor theorem says that if  $f(a) = 0$ , then  $x - a$  is a factor of  $f(x)$ .

**EXAMPLE:** A function is given as  $f(x) = x^3 - 2x^2 - 5x + 6$ .

- a) Show that  $(x + 2)$  is a factor of  $f(x)$

Let  $(x + 2) = 0$ ;

$$f(x) = x^3 - 2x^2 - 5x + 6$$

$$f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6 = -8 - 8 + 10 + 6 = 0$$

$$x = -2$$

Thus  $(x + 2)$  is a factor.

- b) Divide  $f(x) = x^3 - 2x^2 - 5x + 6$

by the factor  $x + 2$

$$\begin{array}{r} x^2 - 4x + 3 \\ x+2 \overline{) \cancel{x^3} - 2x^2 - 5x + 6} \\ \underline{-(\cancel{x^3} + 2x^2)} \phantom{+ 6} \\ -4\cancel{x^2} - 5x + 6 \\ \underline{-(-4x^2 - 8x)} \phantom{+ 6} \\ 3\cancel{x} + 6 \\ \underline{-(3x + 6)} \phantom{+ 6} \end{array}$$

Notice there is no Remainder

- c) Factorize  $f(x)$  completely

$$x^2 - 4x + 3 = (x - 3)(x - 1)$$

$$\therefore (x + 2)(x - 3)(x - 1)$$

### EXERCISE 21:

#### [Multiple Choice Questions]

1. The value of  $k$  for which  $(x - 2)$  is a factor of  $f(x) = x^3 - 3kx + 10$  is equal to:  
A.  $-2$     B.  $-\frac{1}{3}$     C.  $2$     D.  $3$
2. If  $(x - 1)$  is a factor of  $x^3 + kx^2 + x - 6$ , then the value of  $k$  is:  
A.  $1$     B.  $4$     C.  $5$     D.  $6$
3. The expression which is a factor of  $x^3 + 2x^2 - 11x - 12$  is:  
A.  $(x - 3)$     B.  $(x - 1)$     C.  $(x + 2)$     D.  $(x - 2)$

#### [Questions requiring working]

4. Find the value of  $c$  if  $x + 2$  is a factor of  $x^3 + 5x^2 - x + c$ . If  $(x - 1)$  is a factor of  $x^3 + kx^2 - 5x + 6$ , find the value of  $k$ .
5. A function  $f(x)$  is given as  $f(x) = x^3 - 6x^2 + 11x - 6$ .
  - (i) Find  $f(3)$
  - (ii) Use the information in part(i) to factorize  $f(x)$  completely.
6. A function  $f(x)$  is given as  $f(x) = x^3 + x^2 - 17x + 15$ 
  - i) Find  $f(1)$
  - ii) Factorize  $f(x)$  completely
7. A polynomial function is given by  $f(x) = x^3 - kx^2 + x + 6$ .
  - i) Find the value of  $k$  if  $(x + 1)$  is a factor of  $f(x)$
  - ii) Find the other 2 factors.
8. A polynomial function is given by  $f(x) = x^3 + qx^2 - 5x - 6$ .
  - i. Find the value of  $q$  if  $(x + 3)$  is a factor of  $f(x)$ .
  - ii. Hence, factorize  $f(x)$  completely.
9. For a polynomial  $p(x) = x^3 - 4x^2 + x + 6$  it is known that  $p(-1) = 0$ ,  $p(0) = 6$ ,  $p(2) = 0$ ,  $p(3) = 0$ . Write  $p(x)$  in factored form.

**Lesson of Life:** *The man ignorant of mathematics will be increasingly limited in his grasp of the main forces of civilization.* ~John Kemeny

## SUB – STRAND 2.3

## Sequence and Series

$a_1, a_2, a_3, a_4, a_5, \dots, a_n, \dots$

Each of these is called a "term."

This is called the  **$n$ th** term.

Are These Sequences  
Arithmetic or Geometric?

a) -6, 1, 8, 15, 22

+7 +7 +7 +7

Common Ratio? No

Common Diff? Yes

Check: **Arithmetic**

b) 324, 108, 36, 12, 4

$\times \frac{1}{3}$   $\times \frac{1}{3}$   $\times \frac{1}{3}$   $\times \frac{1}{3}$

Common Ratio? Yes

Check:

### CONTENT LEARNING OUTCOMES

- Sigma Notation
- Sequence defined recursively
- Applications of Arithmetic sequence
- Applications of Geometric sequence including Sum to infinity

## SIGMA NOTATION (SYMBOL $\Sigma$ )

**Note:**  $\Sigma$  means to find the sum of terms.

$\sum_{\text{first value}}^{\text{last value}} k$  means to find the sum of all the terms starting from the first value until the last value.

e.g.  $\sum_2^5$  means to find the sum of all the terms starting from the 2<sup>nd</sup> term until the 5<sup>th</sup> term.

**Example:** Evaluate  $\sum_1^4 n^2 + 1$

This means find the sum of all the terms starting from the 1<sup>st</sup> until the 4<sup>th</sup> term using the formula  $n^2 + 1$

i.e.

$$\begin{array}{c|c|c|c} T_1 = 1^2 + 1, & T_2 = 2^2 + 1, & T_3 = 3^2 + 1, & T_4 = 4^2 + 1 \\ T_1 = 1, & T_2 = 5 & T_3 = 10 & T_4 = 17 \end{array}$$

$$\text{Sum: } 1 + 5 + 10 + 17 = 33 \quad \therefore \sum_1^4 n^2 + 1 = 33$$

Or may use tables

| $n$      | $n^2 + 1$      |
|----------|----------------|
| 1        | $1^2 + 1 = 2$  |
| 2        | $2^2 + 1 = 5$  |
| 3        | $3^2 + 1 = 10$ |
| 4        | $4^2 + 1 = 17$ |
| $\Sigma$ | 33             |

Therefore the sum of all these terms is 33.

**EXERCISE 22:** Evaluate the following:

1.  $\sum_{n=2}^5 (-3n^2 + 2)$

2.  $\sum_{x=1}^5 (-2x + 1)$

3.  $\sum_{n=2}^4 \left( \frac{3}{2^n} \right)$

4.  $\sum_{r=3}^6 (2 - r^2)$

## SEQUENCE DEFINED RECURSIVELY

**Note:** **Recursion** is the process of choosing a **starting term** and repeatedly applying the same process to each term to arrive at the next term. Recursion requires that you **know the value of the term immediately before** the term you are trying to find.

A recursive formula always has two parts:

1. the starting value.
2. The recursion equation for  $T_n$  as a function of  $T_{n-1}$  (the term before it.)

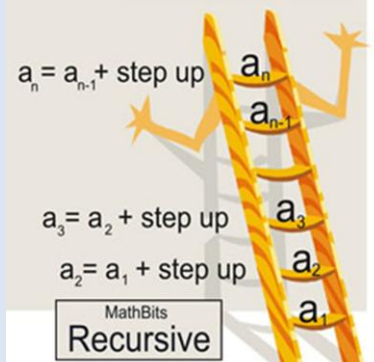
[The term after the  $n^{\text{th}}$  term is denoted as  $T_{n+1}$ ]



Always 2 parts!

Recursion is described as an "iterative" procedure.

The process of recursion can be thought of as climbing a ladder. Each run on the ladder depends upon stepping on the rung below it. You start on the first rung of the ladder, let's say  $a_1$



**EXAMPLE 1:** A sequence is defined as  $T_n = T_{n-1} + n^2$  and  $T_1 = 3$ . Find  $T_2, T_3, T_4$  and  $T_5$

$$\begin{array}{ll} T_1 = 3 & \text{To find } T_2 - \text{work out the } n \text{ value in the recursive formula} \\ T_n = T_{n-1} + n^2 & \text{let } n \text{ be } 2 \\ T_2 = T_{2-1} + 2^2 & \text{Substitute and solve} \\ T_2 = T_1 + 2^2 = T_1 + 4 = 3 + 4 = 7 \end{array}$$

$$\begin{array}{ll} T_3 = T_{3-1} + 3^2 & \text{let } n \text{ be } 3 \\ T_3 = T_2 + 3^2 & \text{[climb up the ladder]} \\ = T_2 + 9 = 7 + 9 = 16 & \text{Substitute and solve} \\ T_3 = T_2 + 3^2 = T_2 + 9 = 7 + 9 = 16 \end{array}$$

$$\begin{array}{ll} T_4 = T_{4-1} + 4^2 & \text{let } n \text{ be } 4 \text{ [climb up the ladder]} \\ T_4 = T_3 + 4^2 & \text{Substitute and solve} \\ = T_3 + 16 = 16 + 16 = 32 \\ T_4 = T_3 + 4^2 = T_3 + 16 = 16 + 16 = 32 \end{array}$$

$$\begin{array}{ll} T_5 = T_{5-1} + 5^2 & \text{let } n \text{ be } 5 \text{ [climb up the ladder]} \\ T_5 = T_4 + 5^2 = T_4 + 25 = 32 + 25 = 57 & \text{Substitute and solve} \end{array}$$

**EXAMPLE 2:** A sequence is defined as  $T_{n+1} = [T_n]^2 - 3T_n$  and  $T_1 = 2$ . Find the 4<sup>th</sup> term of the sequence.

$$\begin{array}{l|l} T_1 = 2 & T_3 = 10 \\ T_2 = T_{1+1} = [T_1]^2 - 3T_1 & T_4 = T_{3+1} = [T_3]^2 - 3T_3 \\ = 2^2 - 3(2) = 4 - 6 & = (10)^2 - 3(10) = 100 - 30 = 70 \\ = -2 & \end{array}$$

Thus  $T_4 = 70$

### EXERCISE 23:

1. A sequence is defined as  $T_{n+1} = [T_n]^2 - T_n$  and  $T_1 = -2$ . Find the 4<sup>th</sup> term of the sequence.
2. Find the 3rd, 4th and 5th term of each sequence.
  - a)  $t_1 = 5$ ;  $t_n = t_{n-1} + 5$
  - b)  $t_1 = 3$ ;  $t_n = 2t_{n-1} - 1$
  - c)  $t_1 = 7$ ;  $t_n = 3t_{n-1} + n$
3. A sequence is defined as  $T_n = T_{n-1} + 5$  and  $T_1 = -4$ . Write the first four terms of the sequence.

### ARITHMETIC SEQUENCE/ PROGRESSION

**Note:** Recall from Year 11,

$a$  is first term of the sequence

$d$  is common difference from the two consecutive terms, i.e

$$d = T_2 - T_1$$

$n$  is  $n$  number of terms

$T_n$  is the  $n$ th term and the formula is given as

$$T_n = a + (n-1)d$$

$S_n$  is the Sum of the  $n$ th term and the formula is given as

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

**EXAMPLE 1:** A sequence is given as  $\langle \log 2, \log 4, \log 8, \log 16, \dots \rangle$

- a) Find the common difference?
- b) Find the 10<sup>th</sup> term of the sequence?
- c) Calculate  $\sum_{n=1}^6 T_n$ .

### Answers:

a) Common difference

$$d = t_2 - t_1$$

formula

$$t_2 = \log 4, t_1 = \log 2$$

$$d = t_2 - t_1 = \log 4 - \log 2$$

$$= \log \frac{4}{2} = \log 2$$

b) 10<sup>th</sup> term:  $t_{10}$

$$T_n = a + (n-1)d$$

$$T_{10} = \log 2 + (10-1)\log 2$$

$$= \log 2 + 9\log 2$$

$$= 10\log 2$$

c) sum of 6<sup>th</sup> term:  $S_{10}$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_6 = \frac{6}{2}[2 \times \log 2 + (6-1)\log 2]$$

$$= 3[2\log 2 + 5\log 2]$$

$$= 3(7\log 2) = 21\log 2$$

**EXAMPLE 2:** Maika placed \$5.00 into an empty money box on 1<sup>st</sup> January last year. How much did he collect after 20<sup>th</sup> February if he kept on adding 60 c daily from 2<sup>nd</sup> January to 20<sup>th</sup> February?

Treating it as a word problem, take out the main points:

$$a - \$5$$

$$d - \$0.60$$

$$n = 31(\text{Jan}) + 20(\text{Feb}) = 51 \text{ days}$$

Money collected in those days i.e  $t_{51}$

$$T_n = a + (n-1)d$$

$$T_{51} = \$5 + (51-1)\$0.60$$

$$= \$5 + 50 \times \$0.60$$

$$= \$35$$

### EXERCISE 24:

1. How many terms are in the series  $5 + 7 + 9 + \dots + 99 + 101$ ?  
Find the sum of the series  $5 + 7 + 9 + \dots + 99 + 101$ .
2. Logs are stacked in a pile with 24 logs on the bottom row and 15 on the top row. There are 10 rows in all with each row having one more log than the one above it. How many logs are in the stack?

3. Find the sum of all the even integers from 20 to 642.
4. The first term and 6<sup>th</sup> term of an arithmetic sequence are -4 and 21 respectively. Calculate the 5<sup>th</sup> term of the sequence.
5. The first, fifth and last terms of an arithmetic sequence are 3, 23 and 103 respectively.

Find the:

- i) Number of terms in the sequence
  - ii) Sum of all the terms in the sequence
6. The first term and 15<sup>th</sup> term of an arithmetic sequence are -5 and 93 respectively. Find the 4<sup>th</sup> term of the sequence.
  7. An auditorium has 20 seats on the first row, 24 seats on the second row, 28 seats on the third row, and so on and has 30 rows of seats. How many seats are in the theatre?
  8. If the sum of the first  $n$  terms of a sequence is  $2n^2 + 3$ , find the first 2 terms of the sequence.
  9. The first 3 terms of an arithmetic sequence are:  $\langle -2, 1, 4, \dots \rangle$   
Find the:
    - i. 5<sup>th</sup> term
    - ii. Sum of the first 10 terms.
  10. What is the sum of the multiples of 3 between 3 and 999, inclusive?
  11. An arithmetic sequence is defined by  $T(n) = 4n - 1$ . What is  $\sum_{n=1}^{24} T(n)$ ?
  13. A sequence is given as  $\langle 4, 2, 0, -2, \dots \rangle$ 
    - (i) Show that the sequence is Arithmetic?
    - (ii) Find the 20<sup>th</sup> term.
    - (iii) What is the sum of the first 20 terms?
    - (iv) Show that the  $n^{\text{th}}$  term of this sequence is given by  $t_n = -2n + 6$
    - (v) Show that the sum of the first  $n$  terms of the sequence is given by  $S_n = -n^2 + 5n$

14. Find the sum of the odd integers from 25 to 75, inclusive.
15. Find the 3<sup>rd</sup> term of the arithmetic sequence with  $t_6 = 24$  and  $t_{15} = 21$ .
16. What is the common difference of the arithmetic sequence with 6<sup>th</sup> term of  $-56$  and 11<sup>th</sup> term of  $11$ ?
17. The first term of an arithmetic sequence is 8, and the sum of the first 20 terms is 730. Find the common difference.
12. Suppose on the first week of a new year you deposit \$1.00 in an empty piggy bank. On the second week you deposit \$1.50; on the third week you deposit \$2.00; and each week thereafter you deposit \$0.50 more than the previous week.
  - a. What kind of sequence do these deposits generate?
  - b. What amount will you deposit in the 52nd week?
  - c. What is the total in the piggy bank at the end of these 52 weeks?

## GEOMETRIC SEQUENCE/PROGRESSION

**Note:** Recall, the variables  
 $a$  is first term

$r$  is common ratio from the two consecutive terms, i.e  $r = \frac{T_2}{T_1}$

$T_n$  is the  $n$ th term and the formula is given as

$$T_n = ar^{n-1}$$

$S_n$  is the Sum of the  $n$ th term and the formula is given as

$$S_n = \frac{a(1-r^n)}{1-r}$$

**EXAMPLE 1:** A sequence is given as  $\langle \frac{1}{2}, \frac{1}{4}\sqrt{2}, \frac{1}{4}, \frac{1}{8}\sqrt{2}, \dots \rangle$

- a) Show that it is a geometric sequence.
- b) Find the 15<sup>th</sup> term of the sequence?
- c) Calculate the sum of 10<sup>th</sup> term.

- a) The common ratio of the is

$$r = \frac{1}{4}\sqrt{2} \div \frac{1}{2}$$

$$= \frac{1}{4}\sqrt{2} \times 2 = \frac{1}{2}\sqrt{2}$$

- b)  $t_{15}$

$$T_n = ar^{n-1}$$

$$T_{15} = \frac{1}{2} \left( \frac{\sqrt{2}}{2} \right)^{15-1}$$

$$= \frac{1}{256}$$

- c)  $S_{10}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{10} = \frac{\frac{1}{2} \times \left( 1 - \frac{\sqrt{2}^{10}}{2} \right)}{\left( 1 - \frac{\sqrt{2}}{2} \right)}$$

$$= \frac{31}{64} \div \left( \frac{2-\sqrt{2}}{2} \right) = \frac{31}{64} \times \frac{2}{2-\sqrt{2}}$$

$$= \frac{31}{32(2-\sqrt{2})} = \frac{31}{64-32\sqrt{2}}$$

### EXERCISE 25:

- A ball is dropped from a height of 8 m. The ball bounces to 80% of its previous height with each bounce. How high (to the nearest tenth of a metre) does the ball bounce on the fifth bounce?
- Discuss why the following are geometric sequences.
  - $\langle 2, 8, 32, 128, \dots \rangle$
  - $\langle 36, -12, 4, -\frac{4}{3}, \dots \rangle$
  - $\langle 24, 36, 54, 81, \dots \rangle$
- A company is offering a job with a salary of \$30,000 for the first year and a 5% rise each year after that. If that 5% rise continues every year, find the amount of money you would earn in a 40-year career.
- For the sequence  $\langle 6, 12, 24, 48, \dots \rangle$  find the common ratio and the 7<sup>th</sup> term.
- In a geometric sequence, the first term is  $-3$  and the common ratio is  $2$ . If the sum of the  $n^{\text{th}}$  term ( $S_n$ ) is  $-3069$ , find:
  - $n$ , the number of terms
  - $t_n$ , the  $n^{\text{th}}$  term of the sequence
- The first 2 terms of a geometric sequence are  $\sqrt{5}$  and  $5$  respectively. Find the fifth term in the sequence.

7. A geometric sequence is given as  $\langle 125, 225, 405, 729, \dots \rangle$ . find the:
  - a) common ratio
  - b) sum of first 9 terms.
8. In 2005, the number of tourists going to a city was 25000. The number has been increasing by 16% every year since then. Find the total number of tourists who went to the city in 2015.
9. A firm exports tuna to USA. It increases its export by 5% each year. During the first year it exports 1500 tins of tuna. How many tins of tuna does it export in its first 12 years of operation?

## SUM TO INFINITY

**Note:** When a geometric sequence has a common ratio between -1 and 1, (i.e.  $-1 < r < 1$ ), then the term becomes smaller and smaller in size as we move along the sequence.

For instance  $\langle 24, 12, 6, 3, 1.5, 0.75, \dots \rangle$  when we add all these terms we see that we get closer and closer to 48 without ever reaching 48 exactly [Check using calculator].

This particular aspect is known as **sum to infinity**.

The formula which is used to find **sum to infinity** is

$$S_{\infty} = \frac{a}{1-r}$$

**Important Note:** the condition is  $|r| < 1$ , i.e.  $-1 < r < 1$ .

**EXAMPLE 1:** What is the general conditions that best ensures that the sum to infinity exists for the sequence  $\langle a, ax, ax^2, ax^3, \dots \rangle$ ?

$$r = \frac{T_2}{T_1} = \frac{ax}{a} = x$$

Find  $r$  and replace in place of  $r$ :

$$|r| < 1$$

$$|x| < 1$$

Thus  $-1 < x < 1$

**EXAMPLE 2:** Find the sum to infinity for the sequence  $\left\langle 12, -3, \frac{3}{4}, \frac{-3}{16}, \dots \right\rangle$

$$a = 12$$

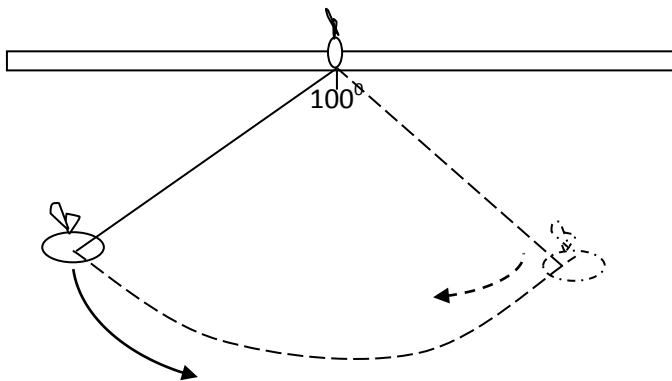
$$r = \frac{-3}{12} = \frac{-1}{4}$$

$$s_{\infty} = \frac{12}{1 - \frac{-1}{4}} = 12 \div \frac{5}{4} = \underline{9.6}$$

[Word Problem: Application of Geometric Series]

**EXAMPLE 3:** Alisi performed an experiment by swinging a pendulum and recording the angle swung by the pendulum in successive swings as shown in the diagram below.

Pendulum swinging  $\langle 100, 90, 81, \dots \rangle$  is geometric sequence.



Her record shows that in the first swing, the pendulum swung at an angle of  $100^{\circ}$ . It swung  $90^{\circ}$  in the second swing,  $81^{\circ}$  in the third and the pattern continues until it comes to rest.

Calculate the:

- angle that the pendulum swung in 8<sup>th</sup> swing.
- sum of the angle swung by the pendulum in the first 8 swings.
- total angle that the pendulum swung through before it comes to rest.

a)  $T_8$

$$r = \frac{T_2}{T_1} = \frac{90}{100} = 0.1$$

$$r = 1 - 0.1 = 0.9$$

Thus

$$T_n = ar^{n-1}$$

$$T_8 = (100)(0.9)^{8-1} = 47.8^{\circ}$$

b)  $S_8$

$$S_n = \frac{a(1-r^n)}{1-r};$$

$$S_8 = \frac{100(1-0.9^8)}{1-0.9}$$

$$= 569.5^0$$

c)  $S_\infty$

$$S_\infty = \frac{a}{1-r} = \frac{100}{1-0.9} = 1000^0$$

## EXERCISE 26:

### [Multiple Choice Questions]

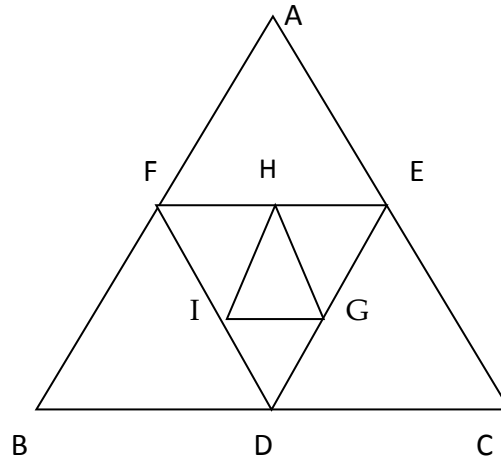
- What is the most general condition that ensures that the sum to infinity exists for the sequence  $2, 2x, 2x^2, 2x^3, \dots$ 
  - $0 < a < 1$
  - $0 < x < 1$
  - $-1 < a < 1$
  - $-1 < x < 1$

### [Questions requiring working]

- Suppose a ball is dropped so that at each bounce, it goes to  $\frac{2}{3}$  of its previous height on each rebound. If it is dropped from a height of 12m, find the total distance travelled by the ball before it comes to rest.
- A pile driver drives a 12 m pole into the ground. Each time it strikes the pole, the pole is driven  $\frac{2}{3}$  the distance it last moved. The first strike drives the pole 3 m into the ground.
  - Determine the distance the pole will be driven by the fifth strike.
  - Use an appropriate formula or otherwise to determine the length of the pole that will be left above the ground no matter how many strikes are made by the pile driver.
- The production from a farm declines as soil fertility drops. The first crop of carrots weigh 60 tonnes. Each year after that, the total weight of carrots produced falls by 5%.
  - Determine the weight of carrots produced in the 3<sup>rd</sup> year.
  - Calculate the total weight of carrots produced after 8<sup>th</sup> year.
  - Calculate the total weight of carrots produced over a long time.

5. A certain amount of yeast is mixed with some dough so that the initial volume of the dough is  $2.80 \text{ m}^3$ . The volume of the dough is then checked after every 10 minutes. It was noticed that during the first 10 minutes, the dough has grown by  $0.64 \text{ m}^3$  &  $0.32 \text{ m}^3$  in the next 10 minutes, & then  $0.16 \text{ m}^3$  during the next 10 minutes & so on.
  - a) Show that the Volume increase forms a geometric sequence
  - b) If the Volume of the dough keeps increasing at that rate until it can't increase anymore, by how much will its Volume increase altogether?
  - c) To what maximum volume will the dough grow?
  
6. A fishing village, in the first year catches 19 683 tonnes of fish. In the next year they catch only one-third of this amount. In each year following that, they only catch one-third of what they caught the year before.
  - (i) Find the amount of fish they catch each year for the first five years.
  - (ii) Calculate the total tonnage caught if they continue to fish until there were no more fish left.
  
7. A hot-air balloon rises 90 m in its first minute of flight. In each succeeding minute, it rises only 90% as far as it did during the preceding minute. What is the final height of the balloon?
  
8. A sequence is given as  $\langle 5, 15, 45, 135, \dots \rangle$ 
  - (i) Is this sequence geometric? Why?
  - (ii) Find the formula which will give the  $n^{\text{th}}$  term of this sequence.
  - (iii) What is the  $6^{\text{th}}$  term?
  - (iv) Explain why the sequence does not have sum to infinity
  
9. A 100L container of toxic waste is buried in a landfill. Each year some of the contents leak into the surrounding water table. The amount leaking, (in litres), each year follows a geometric sequence as shown below.  
 Geometric sequence:  $\langle 15, 12, 9.6, \dots \rangle$ 
  - a) Calculate the amount of toxic waste that leaks over a very long period of time
  - b) How much toxic waste will remain in the container after 10 years?

10. The diagram given below shows triangles inscribed in triangles. Each triangle is an equilateral triangle. Consecutive triangles (inscribed in the previous triangles) are obtained by joining the midpoints as shown below.



Triangle 1 is triangle ABC

Triangle 2 is triangle DEF

Triangle 3 is triangle GHI

The length of each side of Triangle 1 is 16 cm.

Determine:

- The area of Triangle 1
- The area of Triangle 2

The sequence of areas of triangles

$S = \langle \dots \text{area of triangle 1, area of triangle 2, area of triangle 3, } \dots \rangle$  is a geometric sequence.

- Determine the common ratio
- What is the area of the 5<sup>th</sup> triangle?
- What is the sum of the areas of all such triangles

11. A ball is dropped from a height of 8 m. The ball bounces to 80% of its previous height with each bounce. How high (*to the nearest tenth of a metre*) does the ball bounce on the fifth bounce?

**Lesson of Life:** *Mathematics has a pervasive influence on our everyday lives, and contributes to the wealth of the country ~ Anonymous*

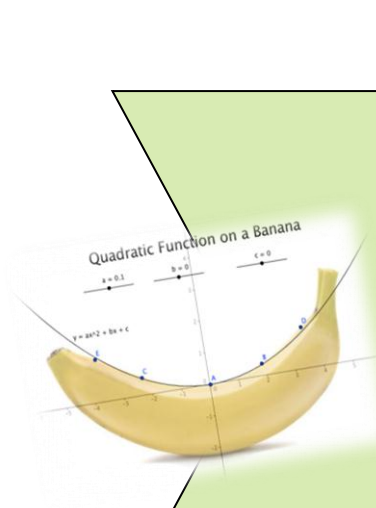
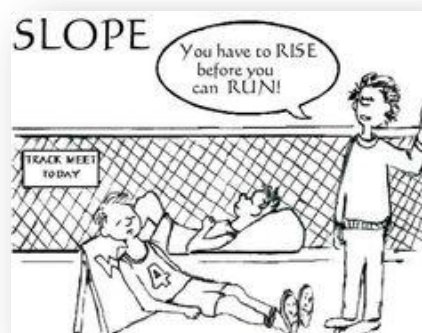
# STRAND THREE

# GRAPHS

**SUB – STRAND 3.1**

***Study and Interpret  
Graphs***

**CONTENT LEARNING  
OUTCOMES**



- Relations and Functions
- Linear Graphs
- Quadratic and it's Turning Point
- Cubic Equations that can be factorized
- Hyperbolic Graphs
- Absolute Value Graph
- Square Root Graph
- Logarithmic and Exponential Functions
- Graph of Circles

## REVISION OF RELATION AND FUNCTIONS

**Note:** Relation has two sets of elements.

**Domain:** First element which are the  $x$  values.

**Range:** Second element which are the  $y$  values.

There are many ways of representing Relation and its Inverse:

1. **Ordered Pair:** write the values of  $x$  and  $y$  in brackets with a comma in between, i.e.  $(x, y)$ .

To find inverse: simply change positions of  $x$  and  $y$ , i.e.  $(y, x)$ .

2. **Equation:** Recall that it will have equal sign. Only new feature is that it includes two variables  $x$  and  $y$ .

For instance,  $y = x + 1$ ,  $3x + y = 6$ ,  $y = x^2$ , etc.

To find inverse: interchange  $x$  and  $y$  variable and make new  $y$  the subject.

$$y = x + 2$$

$$x = y + 2$$

$$x - 2 = y + 2 - 2$$

$$\therefore y^{-1} = x - 2$$

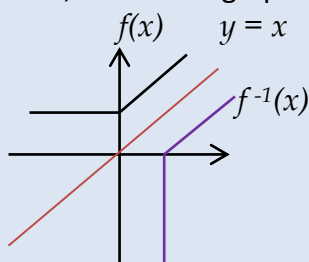
3. **Arrow Diagram:** the starting of the arrow is  $x$  while the finishing arrow sign is  $y$ .

$$x \longrightarrow y$$

To find inverse: simply change the arrow direction,  $x \longleftarrow y$

4. **Graph:** Plot the coordinates and draw the graph.

To find its inverse, reflect the graph in the line  $y = x$ .

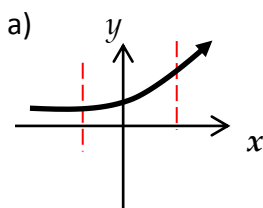


**Function** is where the domain or [ $x$  values/first element] is not repeated.

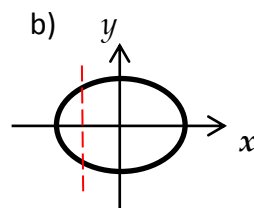
- For graphs, use a vertical line test, i.e. when drawing a vertical line through the graph, the line should cut the graph at only one point. Then it will be a Function, otherwise it's Not Function.

To evaluate  $f(a)$  : since  $x = a$ , substitute  $a$  in place of  $x$  and find the answer, i.e.  $y$  value.

**Example:** Indicate if the following relations are function



Vertical line cuts once,  
therefore it is a function

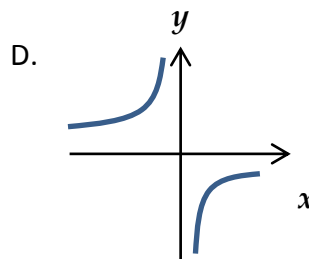
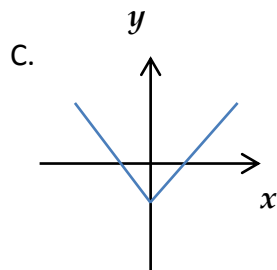
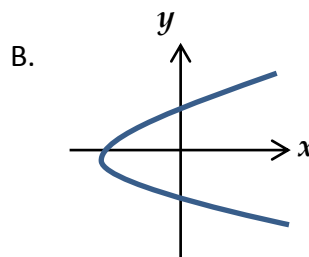
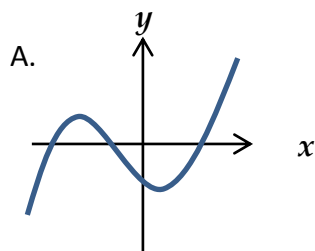


Vertical line cuts more than once,  
therefore it is NOT a function

**Exercise 27:**

**[Multiple Choice Questions]**

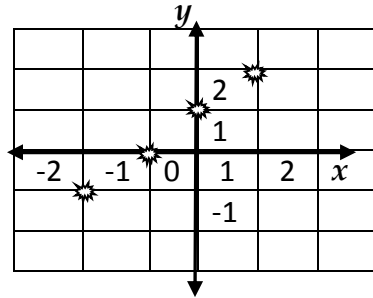
1. Which of the following is not function?



**[Questions requiring working]**

2. The domain of a relation is given as  $\{x : 3 \leq x < 7, x \in \mathbb{R}\}$ .  
What is the range of the inverse relation?
3. A relation is given by  $(x, 2x)$  where  $x \in \{-2, -1, 0, 1, 2\}$ 
  - i). List the range
  - ii). List the relation as a set of ordered pairs.
  - ii). Write the domain of inverse relation.

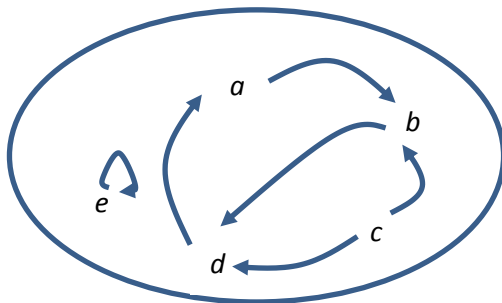
4. A relation is represented by the graph shown below.



- List the domain and range.
  - Is this relation a function?
5. An equation is given as  $y = 3x + 1$  for  $x \in \{-1, 0, 1\}$

- Solve for  $y$  – values
- Write as ordered pairs
- List the domain and range
- Plot the points on the Cartesian plane.
- Write ordered pairs of the inverse
- Is the inverse relation a function?
- Find inverse of the equation
- Draw the graph of the inverse.

6. A relation is shown below:



- List the ordered pairs
- Is this relation a function?
- Draw the inverse of the relation.

**Lesson of Life:** *The difference between a mathematician and a politician is that a mathematician tries to say the most in the least number of words, the politician is the opposite* ~Unknown

## REVISION OF LINEAR GRAPHS

**Note:** The word **Linear** itself tells us that it is derived from the word “Line”.

In this topic, we are going to look at how to draw straight line graphs.

There are **two Methods**, of which intercept method will be used more often.

- Method 1: **Using Tables**
- Method 2: **Intercepts** [To solve for  $x$  – intercept, let  $y=0$  & to solve for  $y$  – intercept, let  $x=0$ . Plot the intercepts and draw a straight line passing through the 2 points].

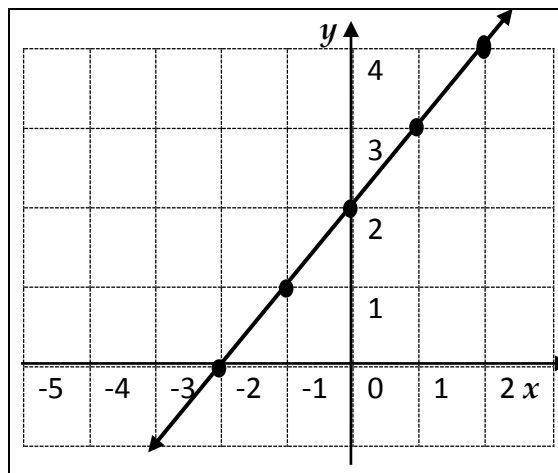
| GRAPH $y = mx + c$                                                                | POSITIVE GRADIENT                                                                 | NEGATIVE GRADIENT                                                                  |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|------------------------------------------------------------------------------------|
| Linear / straight line<br>Example: $x = -1, y = 3,$<br>$3x + 2y = 6, y = -2x + 1$ |  |  |

**EXAMPLE 1:** Sketch the graph of  $y = x + 2$

**Using tables** Take some  $x$  – values, i.e, positive and negative numbers.

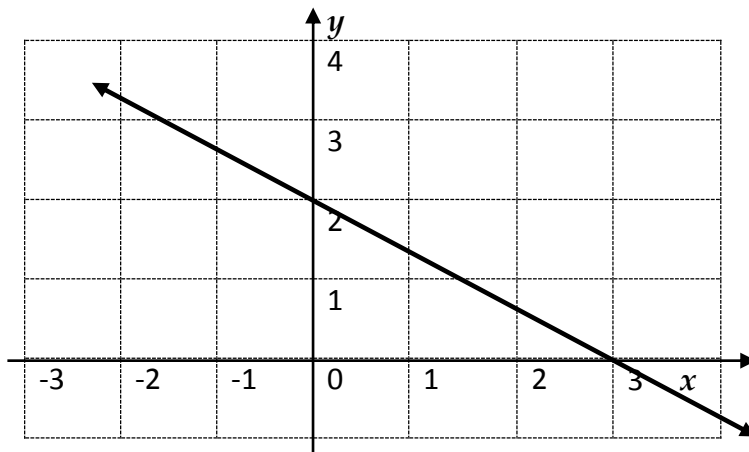
Substitute those  $x$  values to find  $y$

| $x$ | $y = x + 2$  | $(x, y)$  |
|-----|--------------|-----------|
| -2  | $-2 + 2 = 0$ | $(-2, 0)$ |
| -1  | $-1 + 2 = 1$ | $(-1, 1)$ |
| 0   | $0 + 2 = 2$  | $(0, 2)$  |
| 1   | $1 + 2 = 3$  | $(1, 3)$  |
| 2   | $2 + 2 = 4$  | $(2, 4)$  |



**EXAMPLE 2:** Sketch the graph of  $3y + 2x = 6$

**Using Intercepts**



To find  $x$  – int, let  $y = 0$ :

Substitute 0 in place of  $y$  & solve

$$3(0) + 2x = 6$$

$$2x = 6$$

$$\therefore x - \text{int} = 3$$

To find  $y$  – int, let  $x = 0$ :

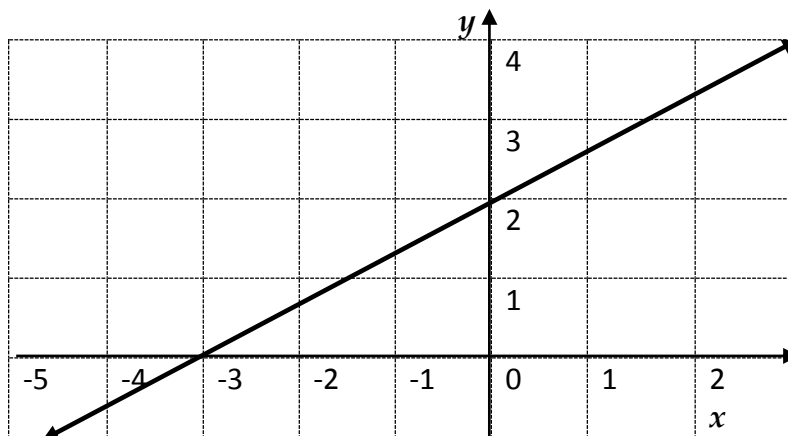
Substitute 0 in place of  $x$  & solve

$$3y + 2(0) = 6$$

$$3y = 6$$

$$\therefore y - \text{int} = 2$$

**EXAMPLE 3:** What is the equation of the graph given below?



The General formula is  $y = m x + c$

First find slope (gradient,  $m$ ):

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{2}{3}$$

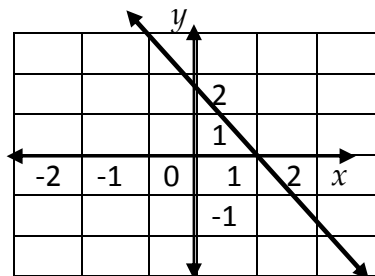
$y$  – int: (cuts at 2 i.e.  $c$  is 2)

Substitute the values in place of the variable  $m$  and  $c$

Thus,  $y = \frac{2}{3} x + 2$

**Exercise 28:**

1. Find the equation of the graph:



2. Sketch the following graphs:

a)  $y = -x + 1$

b)  $x + 2y = 15$

c)  $y = -\frac{1}{3}x + 6$

d)  $y = x + 3$

e)  $3x - 2y = 6$

f)  $y = \frac{1}{2}x - 7$

## QUADRATICS AND IT'S TURNING POINT

**Note:** Recall the quadratic equation [form  $ax^2 + bx + c = 0$ ] and formula  $[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}]$ . Those notes will be very handy here.

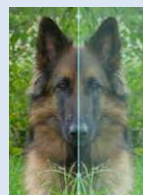
Look at the shape. Did you notice something?

Quadratic graph will have degree 2, i.e. the highest power of 2 for instance,  $y = x^2$ ,  $y = -x^2 + 1$  or  $y = -x^2 + 3x + 1$ .

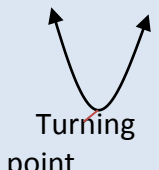
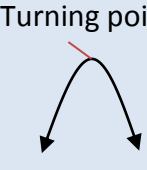
The graph will be **symmetrical** about the **turning point (Vertex)**.

**Symmetrical** objects will have one half as the reflection of the other half.

Line of Symmetry – the line that divides the object into two equal halves.



Turning point – where the graph turns.

| GRAPH $y = \pm a x^2$                         | POSITIVE SHAPE<br>( $y = + a x^2$ )                                                | NEGATIVE SHAPE<br>( $y = - a x^2$ )                                                |
|-----------------------------------------------|------------------------------------------------------------------------------------|------------------------------------------------------------------------------------|
| $y = x^2$ , $y = -x^2 + 2$ ,<br>$y = (x+1)^2$ |  |  |

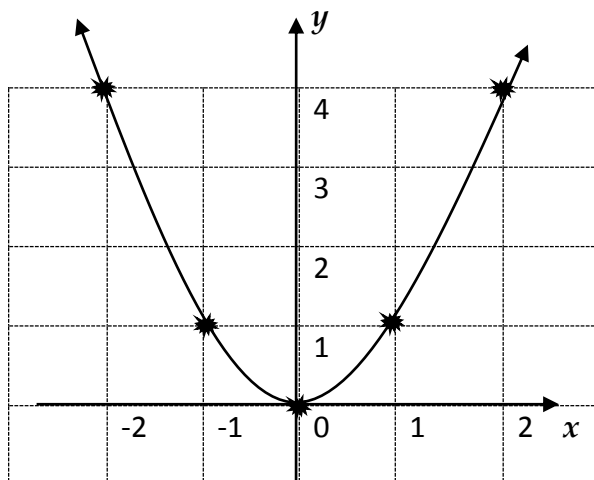
To sketch the graph, follow **intercept** or **table** method from the linear graph. Only new feature is that you may expect two x – intercepts [roots].

**EXAMPLE 1:** Sketch graph of  $y = x^2$

Using tables

Take some x – values, i.e. positive and negative numbers.

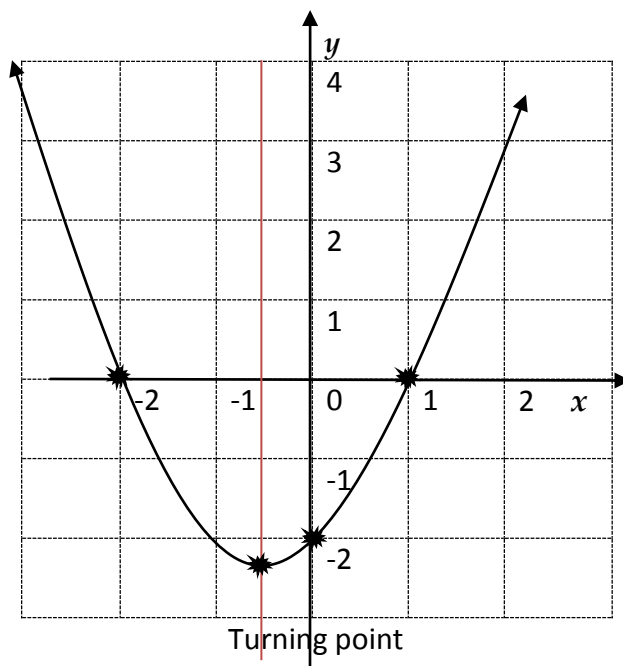
Substitute those x values to find y



| $x$ | $y = x^2$    | $(x,y)$  |
|-----|--------------|----------|
| -2  | $(-2)^2 = 4$ | $(-2,4)$ |
| -1  | $(-1)^2 = 1$ | $(-1,1)$ |
| 0   | $(0)^2 = 0$  | $(0,0)$  |
| 1   | $(1)^2 = 1$  | $(1,1)$  |
| 2   | $(2)^2 = 4$  | $(2,4)$  |

**EXAMPLE 2:** Draw the graph of  $y = x^2 + x - 2$

Using Intercepts



To find x – int, let  $y = 0$ :

Substitute 0 in place of y & solve

$$0 = x^2 + x - 2 \quad \text{Factorize}$$

$$x \quad 2 \quad \rightarrow$$

$$x \quad -1 \quad \rightarrow$$

$$0 = (x + 2)(x - 1) \quad \text{Solve}$$

$$x + 2 = 0, \quad x - 1 = 0$$

$$x = -2, 1 \quad [2 \text{ roots}]$$

To find y – int, let  $x = 0$ :

Substitute 0 in place of x & solve

$$y = x^2 + x - 2$$

$$= 0^2 + 0 - 2$$

$$= -2$$

Shape:  $y = +x^2$

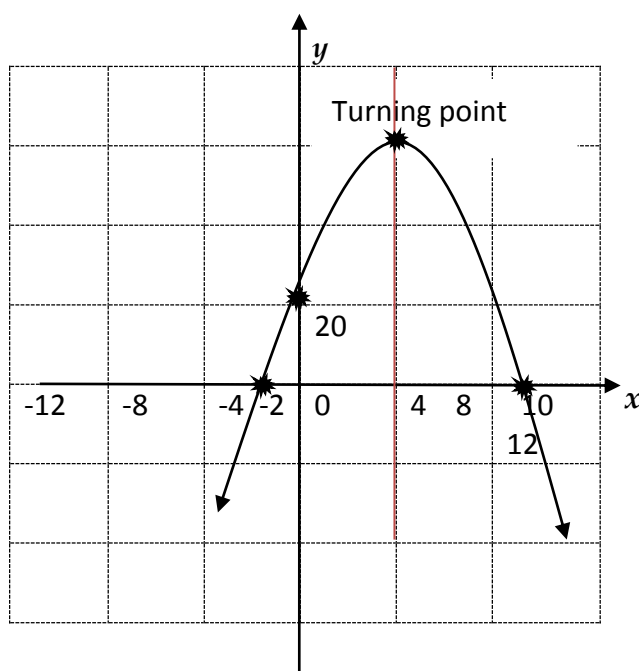
Turning point: Since symmetric, it should be at the midpoint of x – coordinates, thus

$$X_m = \frac{x_1 + x_2}{2} = \frac{-2 + 1}{2} = -\frac{1}{2}$$

$$y = (-0.5)^2 + -0.5 - 2$$



**EXAMPLE 3:** Draw the graph of  $y = -x^2 + 8x + 20$



To find x – int, let  $y = 0$ :

Substitute 0 in place of y & solve

$0 = -x^2 + 8x + 20$  use quadratic formula

$$0 = ax^2 + bx + c,$$

Substitute  $a = -1$ ,  $b = 8$  and  $c = 20$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \frac{-8 \pm \sqrt{8^2 - 4(-1)(20)}}{2(-1)},$$

$$x = \frac{-8 \pm \sqrt{64 + 80}}{-2}, \quad \frac{-8 \pm \sqrt{144}}{-2} = \frac{-8 \pm 12}{-2},$$

$$= \frac{-8 + 12}{-2}, \quad \frac{-8 - 12}{-2},$$

$$x = -2, 10$$

To find y – int, let  $x = 0$ :

Substitute 0 in place of x & solve

$$y = -x^2 + 8x + 20$$

$$= -0^2 + 8(0) + 20$$

$$= 20$$

Shape:  $y = -x^2$

Turning point: Since symmetric, it should be at the midpoint of x – coordinates, thus

$$X_m = \frac{x_1 + x_2}{2} = \frac{-2 + 10}{2} = 4$$

$$y = -4^2 + 8 \cdot 4 + 20 = 36$$



### Exercise 29:

1. Sketch the following graphs:

a)  $y = 375 + 10x - x^2$

b)  $y = x^2 - 25$

c)  $f(x) = (2x - 1)(x - 3)$

d)  $y = -x^2 + 2x + 3$

e)  $f(x) = (2x + 1)(x - 3)$

f)  $y = 3x - x^2$

g)  $f(x) = 8x^2 - 2x - 3$

h)  $f(x) = x^2 - 2x$

i)  $y = -x^2 + 80x - 1500$

j)  $y = -x^2 + 2x + 3$

## REVIEW OF TRANSFORMATION OF QUADRATIC GRAPHS

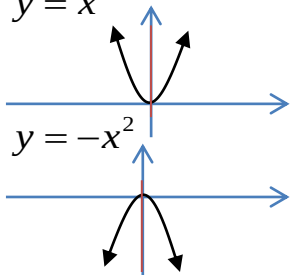
### Note:

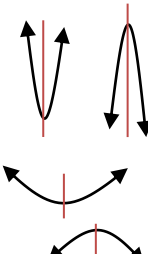
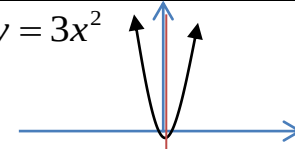
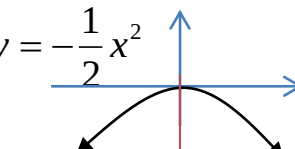
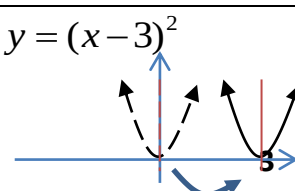
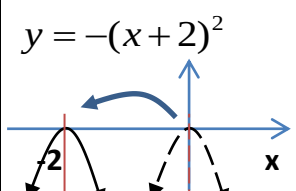
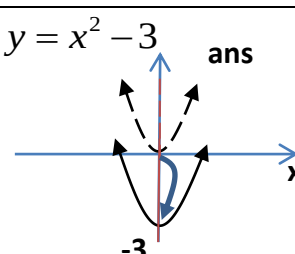
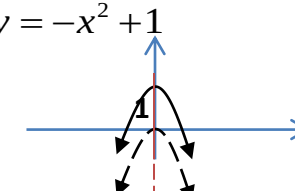
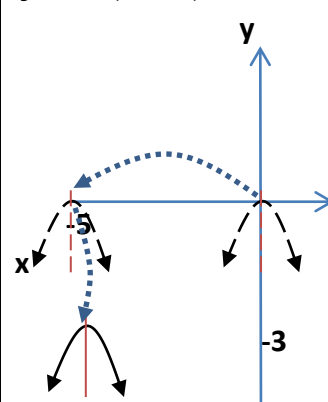
In general, the transformation of Quadratic equation will have the form:

$$y = \pm a(x \pm b)^2 \pm c$$

Shape  $\pm$     Stretching    Shifting along x - axis    Shifting along y - axis

The table below shows how the transformation works by shifting the basic shape:

| General Form                                                                                                                 | Explanation                  | Diagram                                                                                                                                                                                                        | Example                                                                              |
|------------------------------------------------------------------------------------------------------------------------------|------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|
| $y = \pm x^2$<br>This is the basic shape to start with. It will be at the origin (0,0) with y - axis as the line of symmetry | Shape:<br>$+ x^2$<br>$- x^2$ | Positive shape<br><br>Negative shape<br> |  |

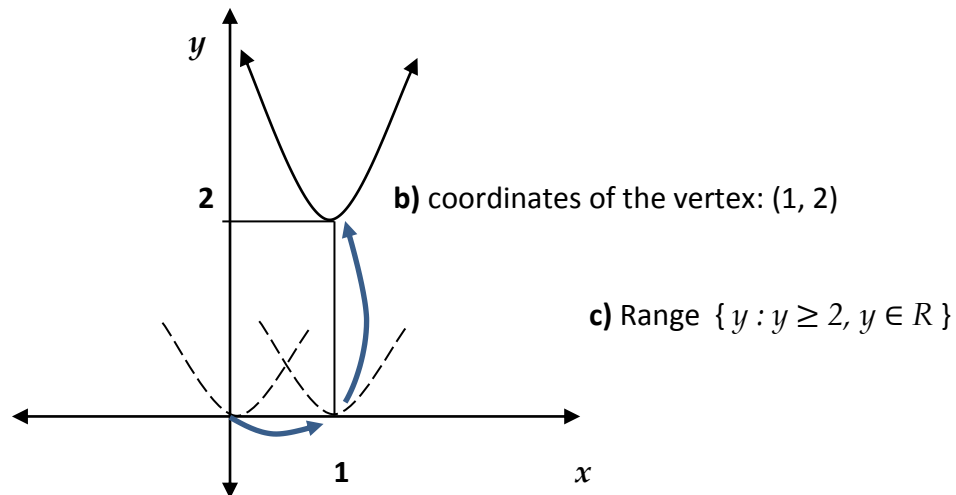
|                                                          |                                                                                                   |                                                                                                      |                                                                                                                                                                                                                           |
|----------------------------------------------------------|---------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $y = \pm ax^2$<br>In front i.e. the coefficient of $x^2$ | Stretching:<br>$-1 > a$<br><i>and</i> $a > 1$<br><i>Narrow</i><br><br>$-1 < a < 1$<br><i>Wide</i> |                     | $y = 3x^2$<br><br>$y = -\frac{1}{2}x^2$<br>           |
| $y = (x \pm b)^2$                                        | Shifting along x-axis:<br>$+b$<br>move left<br><br>$-b$<br>move right                             | Horizontal line<br> | $y = (x-3)^2$<br><br>$y = -(x+2)^2$<br>               |
| $y = x^2 \pm c$                                          | Shifting along y-axis:<br>$+c$<br>move up<br><br>$-c$<br>move down                                |                    | $y = x^2 - 3$ <b>ans</b><br><br>$y = -x^2 + 1$<br> |
| $y = \pm a(x \pm b)^2 \pm c$                             | Combination of all shifting                                                                       | 5 unit left<br>and 3 unit down                                                                       | $y = -(x+5)^2 - 3$<br>                                                                                                                |

**EXAMPLE 1:** The equation is given as  $y = (x - 1)^2 + 2$ .

- (a) Sketch the graph. Clearly show the intercepts
- (b) Give the coordinates of the vertex
- (c) State the range of the function

**Answers:**

a)



**EXAMPLE 2:** The quadratic equation  $f(x) = ax^2 + bx + c$  has  $b^2 - 4ac = 0$  and  $a < 0$ . It also has positive  $x$  - intercept. Sketch the graph that best represents  $f(x)$ .

$a < 0$   
yields

$$f(x) = -x^2 - bx - c$$

this means that  $a$  is negative so replace with negative sign

negative shape

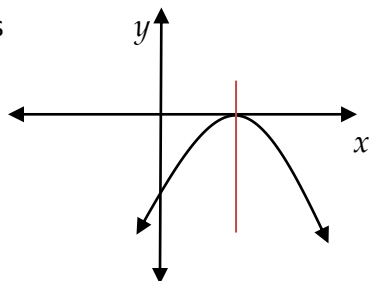


$$b^2 - 4ac = 0$$

axis

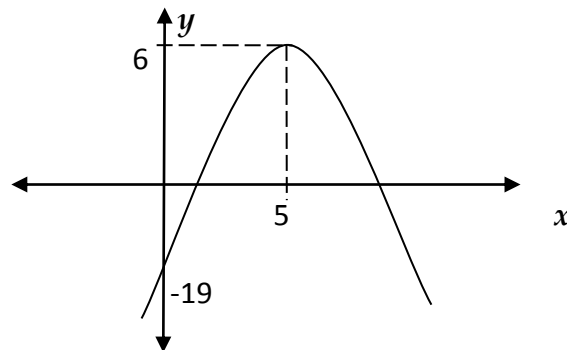
Recall nature of roots (one real root) i.e touching the  $x$  -

Thus



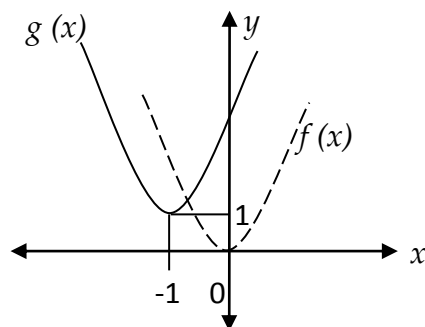
**Exercise 30:**

1. A function is given as  $f(x) = x^2 - 2x$ 
  - i) Find the x and y intercept of the graph of  $f(x)$ .
  - ii) Sketch the graph of  $f(x)$  showing the intercepts and turning points.
  - iii) A second function is given as  $g(x) = f(x+1)$ . On the same pair of axis, Sketch the graph of  $g(x)$ .
  - iv) Name the transformation that maps the graph of  $f(x)$  to  $g(x)$ .
  
2. The diagram given below shows the graph of  $y = f(x)$



Find the equation of the graph of  $f(x)$

3. The graph of  $f(x)$  has been transformed to  $y = g(x)$  and the resulting graph is shown. Find the most appropriate equation for  $g(x)$ .



4. The equation is given as  $y = -(x+2)^2 - 1$ .
  - (a) Sketch the graph. Clearly show the intercepts
  - (b) Give the coordinates of the vertex
  - (c) Identify the axis of symmetry.
  - (d) State the range of the function

## CUBIC EQUATIONS THAT CAN BE FACTORIZED

**Note:** Cubic is derived from the word 'cube' which means power of 3 in algebra, i.e. the highest power of 3.

| GRAPH                  | POSITIVE SHAPE $y = + a x^3$                                                      | NEGATIVE SHAPE $y = - a x^3$                                                       |
|------------------------|-----------------------------------------------------------------------------------|------------------------------------------------------------------------------------|
| Cubic function / graph |  |  |

To sketch the graph, follow **intercept** method from the linear and quadratics graph. Only new feature is that you will expect three x – intercepts [roots].

Also, if you see **square** then the point would be the vertex.

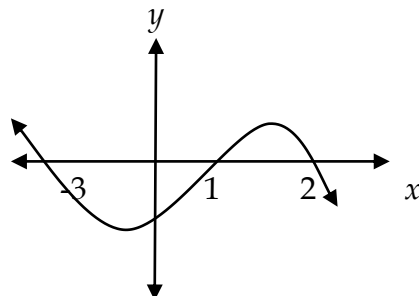
The **transformation of cubic** is exactly the same as quadratic transformation.

In general, the transformation of Cubic equation will have the form:

$$y = \pm a(x \pm b)^3 \pm c$$

Shape  $\pm$ 
Stretching
Shifting along x – axis
Shifting along y – axis

**EXAMPLE 1:** Find the equation of the graph given below



The x – intercepts are given as  $x = -3$ ,  $x = 1$  and  $x = 2$ .

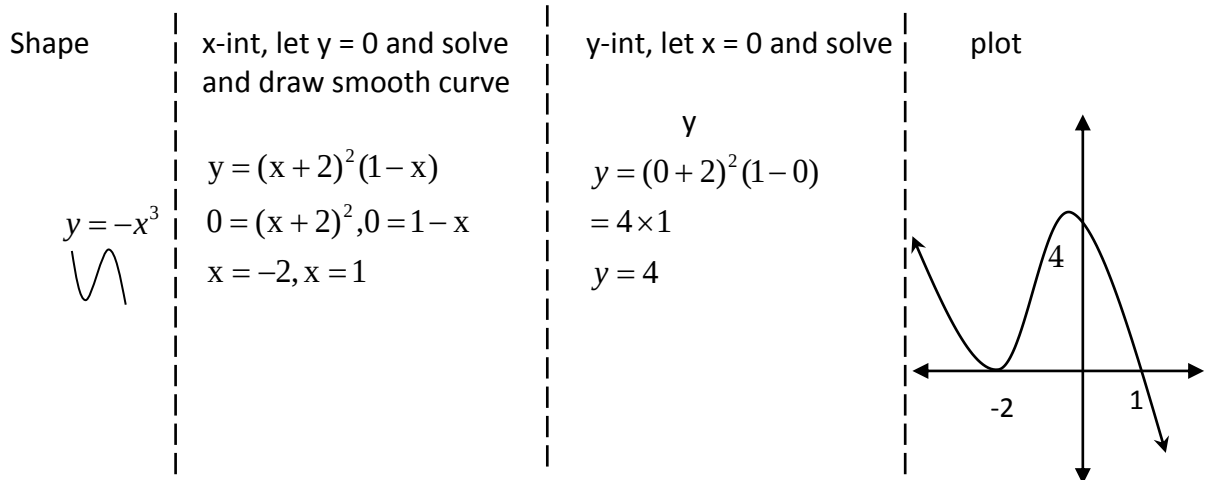
Take it on the left side with the x

$$x = -3, \quad x = 1 \quad x = 2$$

$(x + 3)(x - 1)(x - 2)$  the shape is of negative coefficient so put a negative sign

Thus,  $y = -(x - 2)(x - 1)(x + 3)$

**EXAMPLE 2:** Sketch the graph of  $y = (x + 2)^2(1 - x)$ , show all the intercepts clearly.

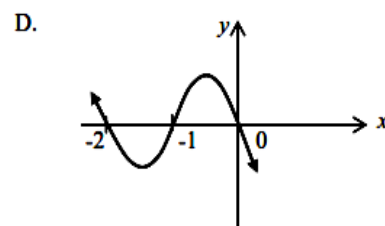
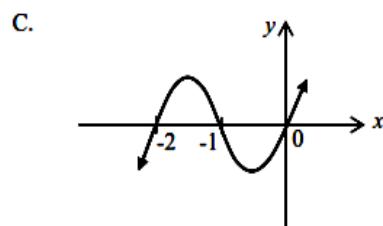
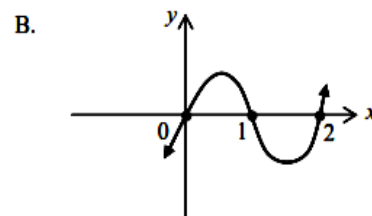
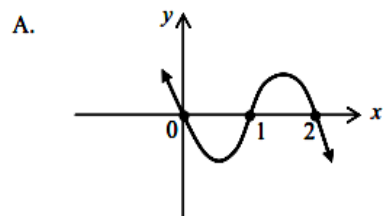


The repeated factor in  $y = (x + 2)^2(1 - x)$  that is  $(x + 2)^2$  means the graph turns at  $x = -2$  as shown above

### EXERCISE 31:

#### [Multiple Choice Questions]

1. Which of the following has the equation  $x(x - 1)(x - 2)$ ?



#### [Questions requiring working]

2. Sketch the following graphs:

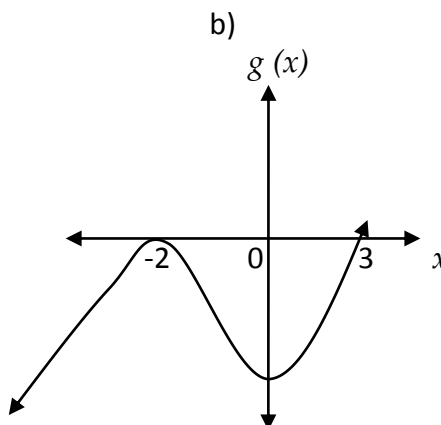
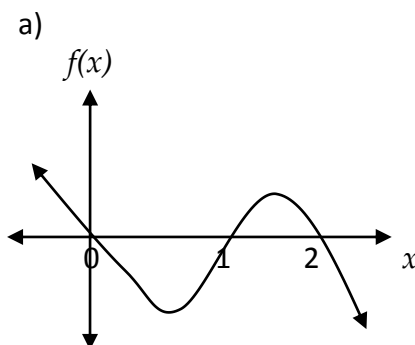
a)  $y = x(x^2 - 1)$

b)  $y = (x + 1)(x - 3)^2$

c)  $y = (3 - x)(x - 2)(x + 1)$

d)  $y = x^2(x + 2)$

3. Write the equation of the following graphs

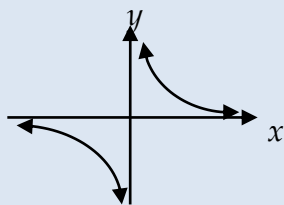


## HYPERBOLIC GRAPHS

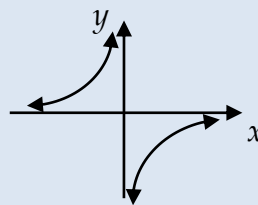
**Note:** Hyperbola will have a **basic form** of  $xy = c$ , where  $c$  is a constant number.

Making  $y$  the subject yields:  $y = \frac{c}{x}$

if  $c$  is positive( draw in quadrants 1 and 3)



if  $c$  is negative(draw in quadrants 2 and 4)



- For the **general form**  $y = \frac{ax+b}{cx+d}$ , important feature to note is the **asymptotes**.

Asymptotes are those lines that you will never cross. In the above diagrams, the vertical asymptote is at  $y = 0$  while the horizontal asymptote is at  $x = 0$ .

- To find the **vertical asymptote**: since denominator of rational function cannot be zero,

let *Denominator* = 0, i.e.  $cx + d = 0$  and Solve

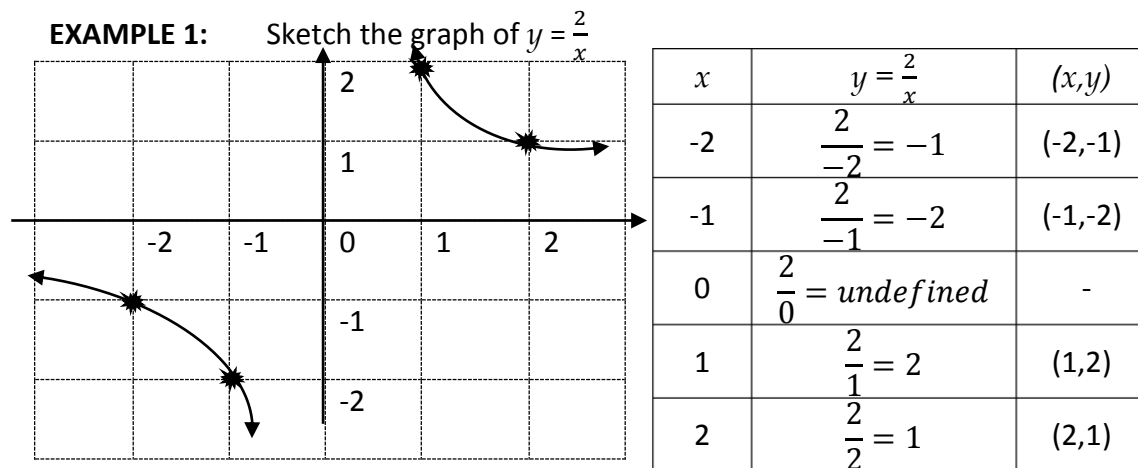
- To find the **horizontal asymptote**: Divide the coefficients of the variable  $x$ ,

$$\text{i.e. } y = \frac{ax}{cx} \text{ or } \frac{a}{c}$$

- To sketch the graph, find the **intercepts**.

If the equation is **not in general form**, recall algebra i.e. **make denominator the same**.

- All hyperbolas have two distinct **branches, directly opposite each other**.
- The branches of hyperbolas get very close to particular lines called the **asymptotes**, without ever actually touching it.

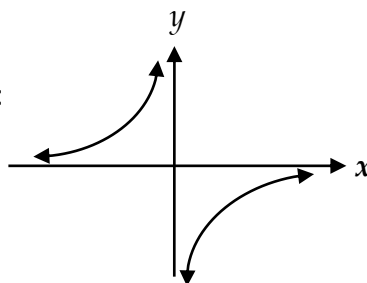


**Note:** The  $y$  value never reach zero because there is no number which will divide into 2 to give zero. Therefore, the  $x$ -axis is an asymptote. Similarly the  $y$ -axis is an asymptote because the  $x$  value never reaches 0.

**EXAMPLE 2:** Sketch the graph of  $y = -\frac{5}{x}$

Making  $y$  the subject:  $y = -\frac{5}{x}$

The shape of negative graph is:



**EXAMPLE 3:** A function is given as  $g(x) = 2 - \frac{3x}{x-1}$

i) Express  $g(x)$  in the form  $g(x) = \frac{ax+b}{cx+d}$

ii) Give the equations of the vertical and horizontal asymptotes.

iii) Sketch the graph of  $g(x)$ , showing clearly all intercepts and asymptotes.

## Answers:

i. Express  $g(x)$

Make denominator the same

$$g(x) = 2 - \frac{3x}{x-1}$$

$$= \frac{2 \cancel{x} - 3x}{1 \cancel{x} - 1}$$

Cross multiply

$$= \frac{2(x-1) - 3x(1)}{(x-1)}$$

Distributive law

Why?

and simplify

$$= \frac{2x - 2 - 3x}{(x-1)}$$

$$\therefore g(x) = \frac{-x-2}{x-1}$$

ii. asymptotes

VA: Let deno. = 0

$$x-1=0$$

$$\therefore x=1$$

HA:  $y =$  coefficient of  $x$

$$y = \frac{-x}{x} = -1$$

iii. Sketch: need intercepts

$y$  - int, let  $x = 0$ , Solve

$$g(x) = \frac{-x-2}{x-1}$$

$$y = \frac{-0-2}{0-1} = \frac{-2}{-1}$$

$$\Rightarrow y = 2$$

$x$  - int, let  $y = 0$ , Solve

$$0 = \frac{-x-2}{x-1},$$

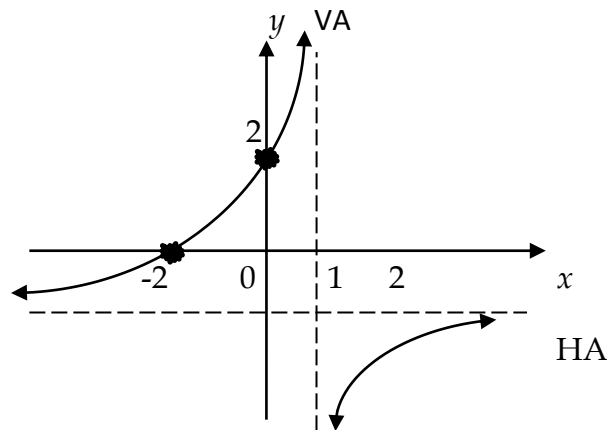
ignore denominator.

$$0 \times (x-1) \text{ is zero}$$

$$0 = -x-2$$

$$x = -2$$

Sketch



## TRANSFORMATION OF HYPERBOLA

**Note:**

- **Vertical transformation:**  $y = \frac{c}{x} \pm a$

where  $+a$ , move the basic graph  $y = \frac{c}{x}$ , 'a' units up

$-a$ , move the graph  $y = \frac{c}{x}$  'a' units down.

- **Horizontal transformation:**  $y = \frac{c}{x \pm b}$

where  $+b$ , move the graph  $y = \frac{c}{x}$  'b' units left or

$-b$ , move the graph  $y = \frac{c}{x}$  'b' units right.

- **Combined Transformation:** In general, the transformation will have the form:

$$y = \frac{1}{\underbrace{x \pm a}_{\substack{\text{Shifting} \\ \text{along } x - \text{axis}}}} \pm \underbrace{b}_{\substack{\text{Shifting} \\ \text{along } y - \text{axis}}}$$

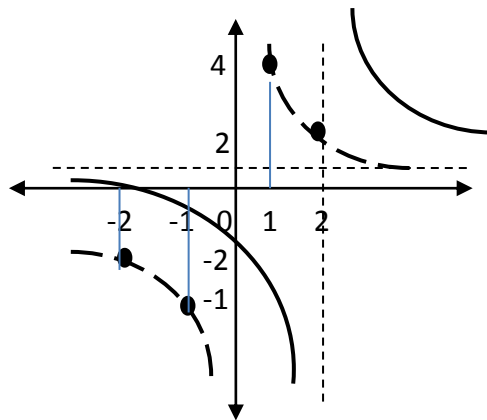
**EXAMPLE 5:** A function is defined as  $f(x) = \frac{4}{x}$ .

- Sketch the graph of  $f(x)$ . Use dotted lines
- A second function  $g(x)$  is defined as  $g(x) = \frac{4}{x-2} + 1$ . Write the equation of the vertical and horizontal asymptotes of  $g(x)$ .
- On the same set of axis used in (i.) above, sketch the graph of  $g(x)$  and show all the intercepts and asymptotes clearly.
- Describe fully the transformation that maps the graph of function  $f(x)$  onto the graph of function  $g(x)$ .

**Answer:**

a) Use tables

| x                    | -2                  | -1                  | 0                            | 1                 | 2                 |
|----------------------|---------------------|---------------------|------------------------------|-------------------|-------------------|
| $f(x) = \frac{4}{x}$ | $\frac{4}{-2} = -2$ | $\frac{4}{-1} = -4$ | $\frac{4}{0} = \infty / und$ | $\frac{4}{1} = 4$ | $\frac{4}{2} = 2$ |



b) First express in the general form:

$$g(x) = \frac{4}{x-2} + \frac{1}{1} \times \frac{x-2}{x-2} = \frac{4+x-2}{x-2} = \frac{x+2}{x-2}$$

VA;  $deno = 0$

$$x-2=0$$

$$x=2$$

HA;  $coefficient$

$$\frac{1x}{1x} = 1$$

$$\therefore y=1$$

c) x-int, let  $y=0$ ;

$$y = \frac{x+2}{x-2}$$

$$0 = \frac{x+2}{x-2}$$

$$x = -2$$

y-int, let  $x=0$

$$y = \frac{x+2}{x-2}$$

$$y = \frac{0+2}{0-2} = -1$$

d) Shifting 1 unit up and 2 units right or translation by vector

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

### EXERCISE 32:

1. Sketch the following graphs, showing all intercepts and asymptotes.

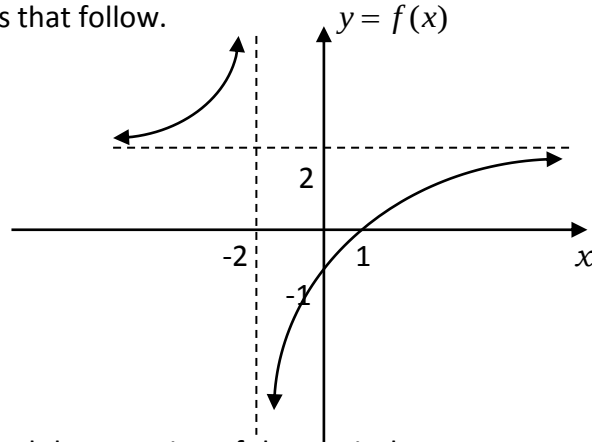
a)  $y = \frac{x-4}{x+2}$

b)  $y = \frac{4x+2}{x-3}$

c)  $y = 1 - \frac{2}{x-3}$

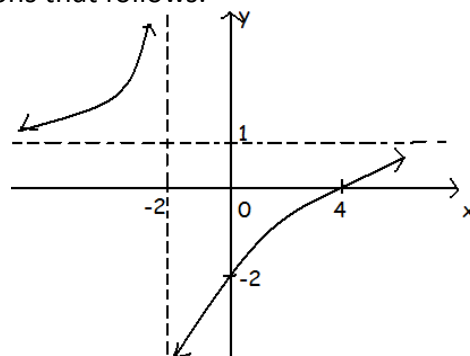
d)  $y = 3 + \frac{1}{x-2}$

2. The graph of the function  $y = f(x)$  is shown below. Use it to answer the questions that follow.



- Find the equation of the vertical asymptote
- State the equation of the horizontal asymptote
- Determine the equation of the function  $y = f(x)$ . Express in the form  $y = \frac{ax+b}{cx+d}$

3. The graph of the function  $y_1 = f(x)$  is shown below. Use it to answer the questions that follows.



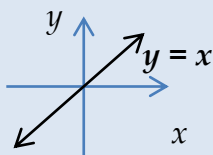
- Find the equation of the vertical asymptote
- State the equation of the horizontal asymptote.
- Determine the equation of the function.
- The graph of the function  $y_2 = g(x)$  is obtained by reflecting the graph of the function  $y_1 = f(x)$  about the y-axis.
- State the equation of the function  $y_2 = g(x)$ .

4. A function is given as  $f(x) = 3 + \frac{5}{x-1}$
- Calculate the x and y intercepts
  - Find the vertical and horizontal asymptotes of  $f(x)$ .
  - Sketch the graph of  $f(x)$ , showing clearly all intercepts and asymptotes.
  - A second function  $g(x)$  is defined as  $g(x) = -f(x)$
  - Describe fully the transformation that maps the graph of function  $f(x)$  onto the graph of function  $g(x)$

## OTHER GRAPHS

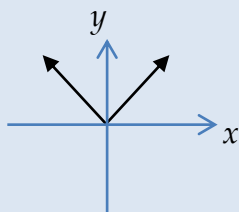
### 1. ABSOLUTE VALUE | |

**Note:** So before we jump straight into graphing  $y=|x|$ , let's first take a look at a graph we already know,  $y=x$ .



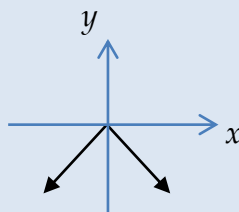
- How is  $y=|x|$  different from  $y=x$ ?  
Well, we know that absolute values don't change positive numbers; they just change the **negative** number and make it **positive**.  
Therefore the negative numbers on the left side will also give positive  $y$ 's.  
The shape of the graph then is:

$$y=|x|$$



[positive shape]

$$y=-|x|$$



[negative shape]

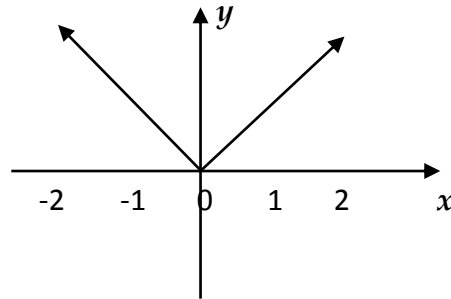
- The transformation will have the form:

$$y = \pm a|x \pm b|^2 \pm c$$

Shape  $\pm$     Stretching    Shifting Along x - axis    Shifting along y - axis

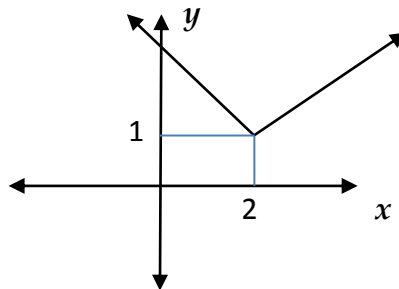
**EXAMPLE 1:** Draw the graph of  $y = |x|$  and State the range

| x         | -2         | -1         | 0         | 1         | 2         |
|-----------|------------|------------|-----------|-----------|-----------|
| $y =  x $ | $ -2  = 2$ | $ -1  = 1$ | $ 0  = 0$ | $ 1  = 1$ | $ 2  = 2$ |



Range =  $\{y: y \geq 0, y \in R\}$

**EXAMPLE 2:** The graph of the function  $f(x)$  is shown below.



Write the equation of  $f(x)$  and state the range.

Shift 2 units right and 1 unit up  $\therefore f(x) = |x - 2| + 1$

$R = \{y: y \geq 1, y \in R\}$

### EXERCISE 33:

1. Sketch the following graphs

a) Graph  $y = |x| + 2$

c) Graph  $y = -|x + 2|$

b) Graph  $y = -|x| + 2$

d) Graph  $y = |x^2 - 3x - 4|$

2. (a) Sketch the graph of  $y = |x| - 1$  and clearly indicate the intercepts

(b) State the domain and range of the function

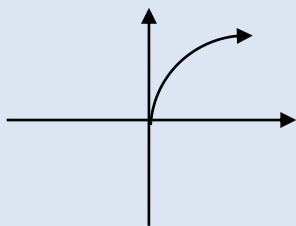
**Lesson of Life:** *The laws of Nature are but the mathematical thoughts of God*  
–Euclid

## 2. SQUARE ROOT GRAPH $\sqrt{\phantom{x}}$

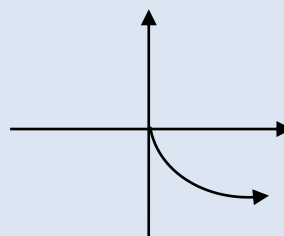
**Note:** The basic Square root graph will have the form  $y = \sqrt{x}$  is same as  $y = x^{\frac{1}{2}}$

- The shape of the graph:

$$y = \sqrt{x}$$



$$y = -\sqrt{x}$$



- The transformation will have the form:

$$y = \pm a\sqrt{x \pm b}^2 \pm c$$

Shape  $\pm$

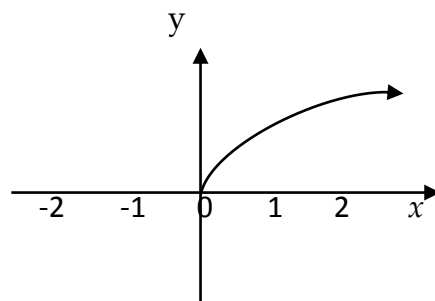
Stretching

Shifting  
along x – axis

Shifting  
along y – axis

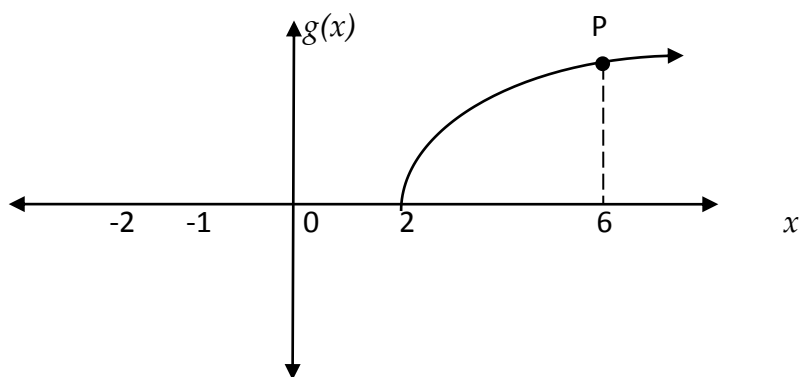
**EXAMPLE 1:** Draw the graph of  $y = \sqrt{x}$  and State the domain

| x              | - 1                            | 0              | 1              | 2                 | 4              |
|----------------|--------------------------------|----------------|----------------|-------------------|----------------|
| $y = \sqrt{x}$ | $\sqrt{-1} = \text{undefined}$ | $\sqrt{0} = 0$ | $\sqrt{1} = 1$ | $\sqrt{2} = 1.41$ | $\sqrt{4} = 2$ |



Domain =  $\{x: x \geq 0, x \in R\}$

**EXAMPLE 2:** The diagram below shows the graph of  $g(x) = \sqrt{x-2}$



- Write down the coordinates of point P.
- Find  $g(-x)$ .
- On the pair of axis drawn, draw the graph of  $g(-x)$ .
- Describe the transformation.
- State the domain and range of  $g(-x)$ .

**Answers:**

- a) coordinates of point P  
 $x = 6$ , *Substitute*

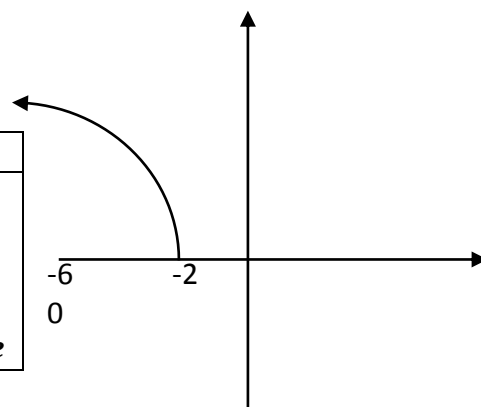
$$g(6) = \sqrt{6-2} \\ = \sqrt{4} = 2 \therefore (6, 2)$$

- b) Substitute  $-x$  in place of  $x$

$$g(x) = \sqrt{x-2} \\ g(-x) = \sqrt{-x-2}$$

- c) Use tables:

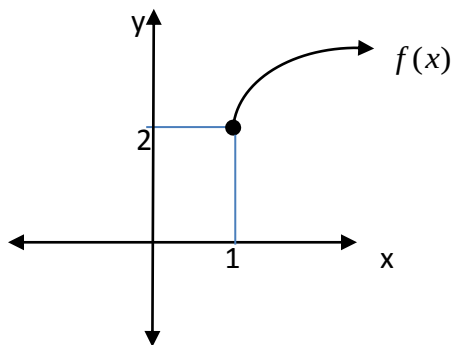
| x                     | -2                                        | -1                                                                       | 0                                                                  |
|-----------------------|-------------------------------------------|--------------------------------------------------------------------------|--------------------------------------------------------------------|
| $g(-x) = \sqrt{-x-2}$ | $g(-(-2))$<br>$= \sqrt{-(-2)-2}$<br>$= 0$ | $g(-(-1))$<br>$= \sqrt{-(-1)-2}$<br>$= \sqrt{-1}$<br>$= \text{undefine}$ | $g(-0)$<br>$= \sqrt{-0-2}$<br>$= \sqrt{-2}$<br>$= \text{undefine}$ |



- d) Reflection in the  $y$  - axis

- e) Domain  $\{x : x \leq -2, x \in R\}$   
 Range  $\{y : y \geq 0, y \in R\}$

**EXAMPLE 3:** The graph of the function  $f(x)$  is shown below.



- a) Write the equation.
- b) State the domain and range of  $f(x)$

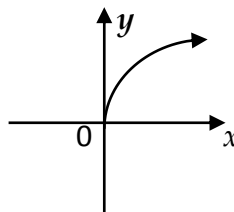
- a) Shifted 1 unit right and 2 units up  
and it's a positive graph

$$y = \sqrt{x-1} + 2$$

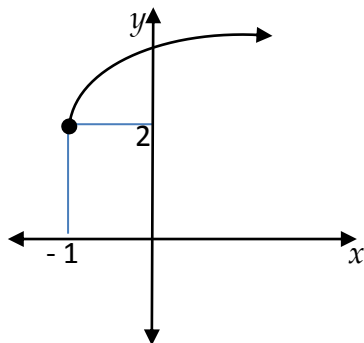
- b)  $D = \{x : x \geq 1, x \in \mathbb{R}\}$   
 $R = \{y : y \geq 2, y \in \mathbb{R}\}$

**EXERCISE 34:**

- State domain and range of the graph:

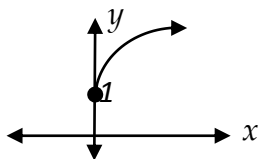


- Sketch the graph of  $y = -\sqrt{x-2}$  and state the domain and range.
- The graph of the function  $g(x)$  is shown below.



- a) Write the equation of  $g(x)$ .
- b) Write down the  $y$ -intercept of  $g(x)$
- c) State the domain and range of  $g(x)$

4. The graph of the function  $f(x)$  is shown below.



- a) What is the equation?
- b) State the domain and the range of  $f(x)$
- c) Reflect the graph on  $y = x$  line.
- d) Write down the equation of the inverse.

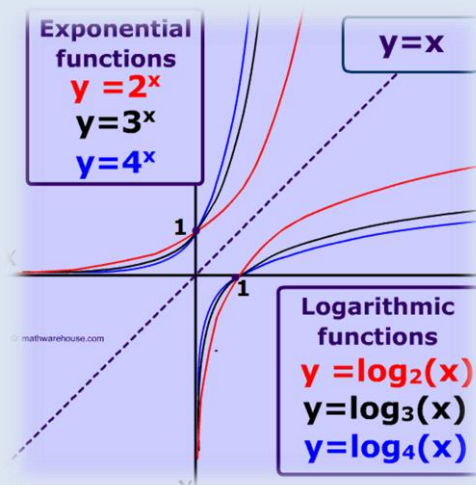
### 3. LOGARITHMIC / EXPONENTIAL FUNCTIONS

**Note:** On your calculator,

You will notice the two functions:  $e^x$  and  $\ln x$  are on the same button because they are **inverses** of each other. If we plot them on the same axes, we see that they are just **reflections** of each other in the line  $y = x$ .

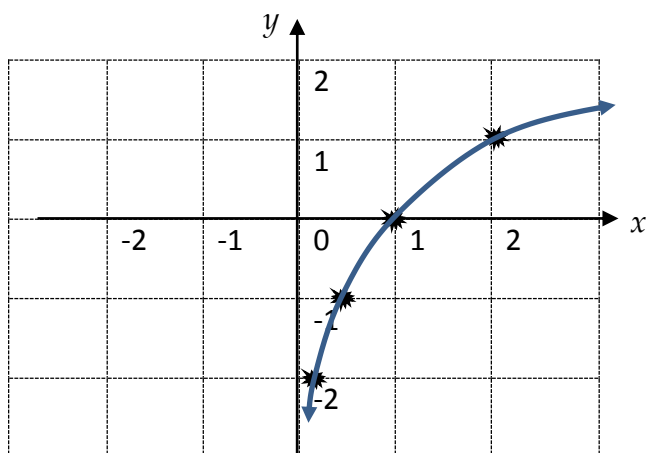
Similarly, if we see the button:  $10^x$  and  $\log$ . These two are inverses of each other. This **log** is of base 10 only. **Don't** use this if you have to find **log of other numbers**. For that case, recall the rules of logarithm.

**Using Tables** [get some  $x$  values, substitute and find  $y$  values, plot points and draw graph]. Recall, anything raised to power zero gives one, thus the  $y$  – intercept of any exponential graph is always 1, (0,1) except for the transformed graph.



**EXAMPLE 1:** Sketch graph of  $y = \log_2 x$

| x                                           | 0.125 | 0.25 | 0.5 | 1 | 2 |
|---------------------------------------------|-------|------|-----|---|---|
| $y = \log_2 x$<br>$= \frac{\log x}{\log 2}$ | -3    | -2   | -1  | 0 | 1 |



- The y-axis is an asymptote because  $\log 0$  is undefined

**EXAMPLE 2:** A function is given as  $f(x) = 3^x$

- Find the coordinates of the y – intercept
- Sketch the graph of  $f(x)$  and label it clearly

Another function is defined as  $g(x) = \log_3 x$

- On the pair of axes, Sketch the graph of  $g(x)$ , showing the x – intercept clearly
- Describe fully the transformation that maps the graph of  $f(x)$  onto the graph of  $g(x)$ .

**Answers:**

- y – int, let  $x = 0$  and solve

$$f(x) = 3^x$$

$$\therefore y = 3^x$$

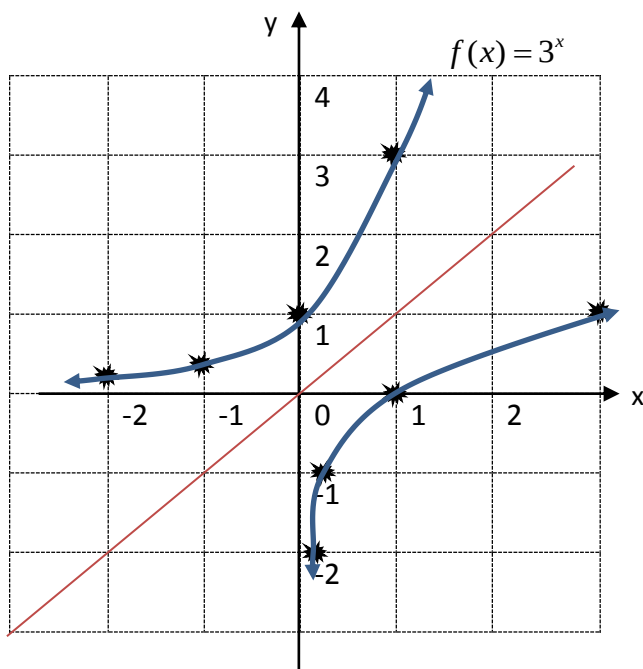
$$= 3^0 = 1$$

$$(0,1)$$

ii. Sketch: using tables

| x         | -2                         | -1                         | 0             | 1             | 2             |
|-----------|----------------------------|----------------------------|---------------|---------------|---------------|
| $y = 3^x$ | $y = 3^{-2} = \frac{1}{9}$ | $y = 3^{-1} = \frac{1}{3}$ | $y = 3^0 = 1$ | $y = 3^1 = 3$ | $y = 3^2 = 9$ |

ii / iii Sketch



Prove that  $f(x) = 3^x$  and  $y = \log_3 x$  are inverses of each other.

To find the inverse:

$$y = 3^x$$

Interchange **x and y**

$$3^y = x$$

$$\log 3^y = \log x$$

$$y = \frac{\log x}{\log 3}$$

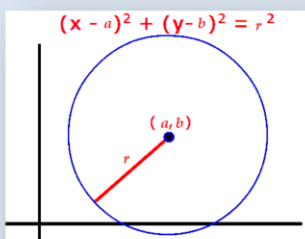
$$y = \log_3 x$$

iv. Reflection in the line  $y = x$ .

## 4. GRAPH OF CIRCLES

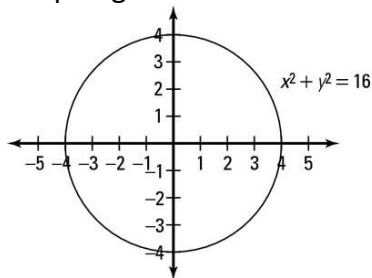
**Note:** The equation of a circle at the centre of origin (0, 0) with the radius,  $r$ , is given as  $x^2 + y^2 = r^2$

- The equation makes a circle as long as the exponents on  $x$  and  $y$  are both 2 (power 2) and the coefficients on  $x$  and  $y$  are both same (usually 1).
- The general equation of a circle with radius  $r$  and centre  $(a, b)$  is  $(x - a)^2 + (y - b)^2 = r^2$



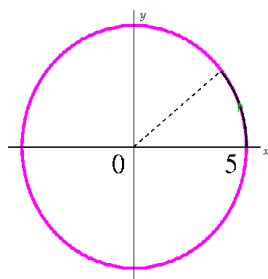
**EXAMPLE 1:** Sketch the graph of  $x^2 + y^2 = 16$

Graphing a circle centered at the origin with  $r^2 = 16$ , thus  $r = 4$ .



**EXAMPLE 2:** Write the equation of the graph

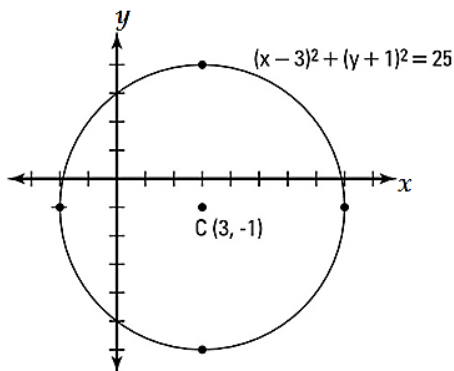
$$r = 5$$



$$x^2 + y^2 = r^2 = 5^2$$

$$\therefore x^2 + y^2 = 25$$

**EXAMPLE 3:** Sketch the graph of  $(x-3)^2 + (y+1)^2 = 25$



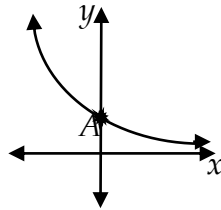
**EXERCISE 35:**

1. The coordinates of the end points of the diameter of a circle are  $(-4, 0)$  and  $(4, 0)$ .
  - a) Write down the equation of this circle.
  - b) Determine the domain and the range

2. A function is given as  $f(x) = 2^x$ .

- a) Sketch the graph of  $f(x)$  and label it clearly
- b) Find the expression for  $g(x) = -f(x)$
- c) Sketch the graph of  $g(x)$  and label it clearly
- d) Describe fully the transformation that maps the graph of  $f(x)$  onto the graph of  $g(x)$ .

3. The diagram below shows the graph of  $y = \frac{1}{2^x}$ .



- a) Give the coordinates of point A.
- b) On the pair of axes, draw the graph of its inverse.

4. Sketch the graph of  $x^2 + y^2 = 9$  and state its domain.

5. Sketch the following circles:

a)  $x^2 + y^2 = 3^2$

b)  $x^2 + y^2 = 36$

c)  $2x^2 + 2y^2 = 8$

d)  $(x-2)^2 + (y+3)^2 = 9$

e)  $3(x+1)^2 + 3(y-2)^2 = 12$

f)  $(x+2)^2 + (y+1)^2 = 14$

6. Sketch the following graphs:

a)  $y = \log x$

b)  $y = \log_3 x$

c)  $y = \log_2(x+1)$

7. Sketch the graph of  $y = \left(\frac{1}{4}\right)^x$  and give the equation of its inverse function.

8. Sketch the graph  $y^2 = 9 - x^2$  and state its domain and range.

**Lesson of Life:** *If there is a God, he's a great mathematician* ~ Paul Dira

**MATHEMATICS OF LIFE**

$$\begin{array}{rcl} \text{LIFE} + \text{LOVE} & = & \text{HAPPY} \\ \text{LIFE} - \text{LOVE} & = & \text{SAD} \\ \hline 2\text{LIFE} & & = \text{HAPPY} + \text{SAD} \end{array}$$
$$\text{LIFE} = \frac{\text{HAPPY} + \text{SAD}}{2}$$
$$\text{LIFE} = \frac{1}{2} \text{HAPPY} + \frac{1}{2} \text{SAD}$$

**CONTENT LEARNING  
OUTCOMES**

- Simultaneous Equations: Applications
- Linear and Quadratic Equation
- Linear and Hyperbolic Equation
- Linear Equation and Circle

## APPLICATIONS OF SIMULTANEOUS EQUATIONS

**Note:** Simultaneously means to solve for variables at the same time.

Three methods include:

- Elimination method
- Substitution
- Graphical method

**Elimination method:**

- To eliminate one variable, either add or subtract the two equations. Remember one of the variables should get eliminated so there may be a need to multiply equations by a constant.
- Solve for one variable first
- Substitute in any equation to find the second value.

**Substitution method** involves the substituting of one equation into another equation. Then solve for the two variables.

**Graphical method** is **not** to be encouraged since a lot of approximation is required in some cases.

**EXAMPLE 1:** Solve these two equations simultaneously:

$$4x - 3y = 3 \text{ and } 10x + 3y = 4$$

$$\begin{array}{r} 4x - 3y = 3 \\ + \quad 10x + 3y = 4 \\ \hline \end{array}$$

$$\begin{array}{l} 14x = 7 \\ x = \frac{1}{2} \text{ or } 0.5 \end{array}$$

Substitute in any equation:

$$\begin{array}{l} 4x - 3y = 3 \\ 4(0.5) - 3y = 3 \\ 2 - 3y = 3 \\ 2 - 2 - 3y = 3 - 2 \\ -3y = 1 \\ y = -\frac{1}{3} \end{array}$$

**EXAMPLE 2:**

A total of 925 tickets were sold for \$5925. If adult tickets cost \$7.50 and children's tickets cost \$3.00, how many tickets of each kind were sold?

**Answers:**

Let  $x$  be the number of adult tickets.

Let  $y$  be the number of children's tickets.

$$\begin{array}{rcl} x + y & = & 925 \\ 7.5x + 3y & = & 5925 \end{array} \quad \text{Multiply this equation by 3 and subtract}$$

$$\begin{array}{r} 3x + 3y = 2775 \\ - \quad 7.5x + 3y = 5925 \\ \hline \end{array}$$

$$\begin{array}{l} -4.5x = -3150 \\ x = 700 \end{array}$$

Substitute in any equation:

$$\begin{array}{l} x + y = 925 \\ 700 + y = 925 - 700 \\ y = 225 \end{array}$$

Thus 700 adult tickets and 225 children's tickets.

**EXERCISE 36:**

- Two numbers have a sum of 90 and one is 5 times the other.
  - Write down the pair of simultaneous equations.
  - Solve for the values of the two numbers.
- The total number of girls and boys in a class is 42. There are more girls than boys in the class. The difference between the number of boys and girls is 16. Find the number of girls and boys in the class.
- The total numbers of mangoes and oranges in a pile is 47. There are more mangoes than oranges in the pile. The difference between the number of mangoes and oranges in the same pile is 17.
  - Use the above information to give a pair of simultaneous equations
  - Find the number of mangoes and oranges
- The cost of book B is \$15.00 more than three times the cost of book A. If the cost of books A and B together is \$75.00, what is the cost of book A?

## POINTS OF INTERSECTION(S)

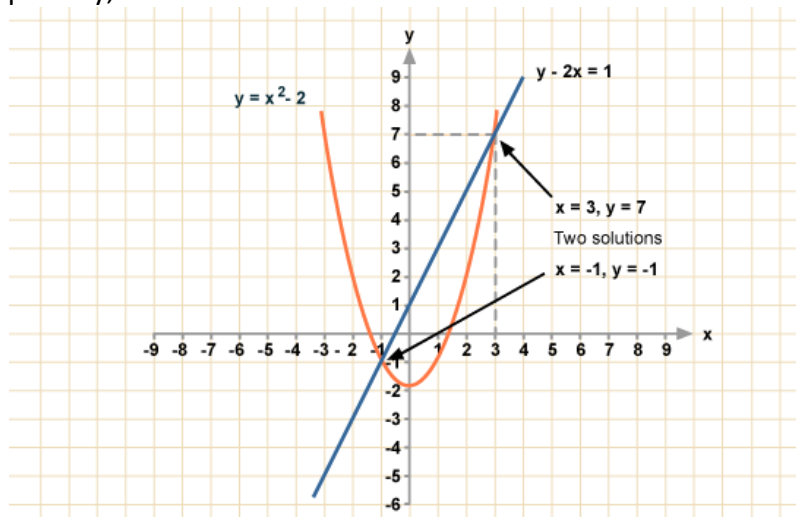
### 1. LINEAR AND QUADRATIC EQUATION

**Note:** Quadratic equation and straight line graphs will meet at, at most two places. The answers to be written in coordinate form:  $(x, y)$ .

You may be required to use factorization technique or quadratic formula.

**EXAMPLE 2:** Find the coordinates of the points of intersection of the parabola  $y = x^2 - 2$  and the straight line  $y - 2x = 1$ .

Graphically,



Using **Substitution method**:

Substitute the first equation in the second one by replacing in place of  $y$ .

$$y = x^2 - 2$$



$$y - 2x = 1$$

$$(x^2 - 2) - 2x = 1$$

Collect all on one side, since it's a trinomial.

$$x^2 - 3 - 2x = 0$$

Rearrange from highest degree

$$x^2 - 2x - 3 = 0$$

Factorise

$$\begin{array}{l} x \quad \nearrow -3 \\ x \quad \searrow 1 \end{array}$$

$$\text{Thus } (x - 3)(x + 1) = 0$$

$$x = -1, 3$$

Substitute the x value in any equation to solve for y:

$$\text{If } x = -1, y = x^2 - 2 = (-1)^2 - 2 = -1$$

$$\text{If } x = 3, y = x^2 - 2 = (3)^2 - 2 = 7$$

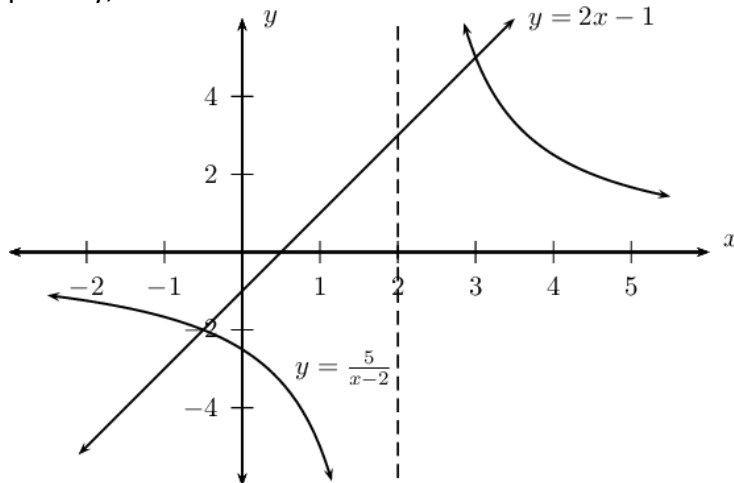
$$\therefore (-1, -1) \text{ and } (3, 7)$$

## 2. LINEAR AND HYPERBOLIC EQUATION

**Note:** Hyperbolic equation and straight line graphs will meet at, at most two places.

**EXAMPLE 5:** Find the coordinates of the points of intersection of the  $y = \frac{5}{x-2}$  and the straight line  $y = 2x - 1$ .

Graphically,



Using **Substitution method**:

Substitute the first equation in the second one by replacing in place of y.

$$y = \frac{5}{x-2}$$



$$y = 2x - 1$$

$$2x - 1 = \frac{5}{x-2}$$

Take denominator on the left side.

$$(2x - 1)(x - 2) = 5$$

Expand the brackets

$$2x^2 - 4x - x + 2 = 5$$

Collect all on one side, since it's a trinomial.

$$2x^2 - 5x - 3 = 0$$

You may use quadratic formula.

$$a = 2, b = -5, c = -3$$

Substitute the values:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)} = \frac{5 \pm \sqrt{25 + 24}}{4} = \frac{5 \pm \sqrt{49}}{4} = \frac{5 \pm 7}{4}$$

$$\Rightarrow x = 3, -\frac{1}{2}$$

Substitute the x value in any equation to solve for y:

$$\text{If } x = 3, y = 2x - 1 = 2(3) - 1 = 5$$

$$\text{If } x = -\frac{1}{2}, y = 2x - 1 = 2(-\frac{1}{2}) - 1 = -2$$

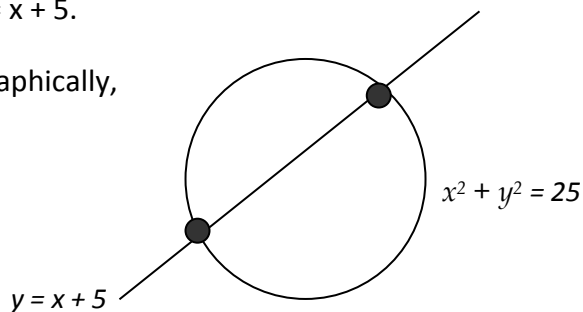
$$\therefore (3, 5) \text{ and } (-\frac{1}{2}, -2)$$

### 3. LINEAR EQUATION AND CIRCLE

**Note:** Circles and straight line graphs will meet at, at most two places. You may be required to use factorization technique or quadratic formula.

**EXAMPLE 6:** Find the *points of intersection* for the equations:  $x^2 + y^2 = 25$  and  $y = x + 5$ .

Graphically,



Using **Substitution method**:

Substitute the first equation in the second one by replacing in place of  $y$ .

$$y = x + 5$$

$$x^2 + y^2 = 25$$

$$x^2 + (x + 5)^2 = 25$$

Expand the brackets

$$x^2 + (x^2 + 10x + 25) = 25$$

$$2x^2 + 10x = 0$$

Factorize

$$2x(x + 5) = 0$$

Solve

$$x = -5, 0$$

Substitute the  $x$  value in any equation to solve for  $y$ :

$$\text{If } x = 0, y = x + 5 = y = 0 + 5 = 5$$

$$\text{If } x = -5, y = x + 5 = y = -5 + 5 = 0$$

$$\therefore (0, 5) \text{ and } (-5, 0)$$

### EXERCISE 37:

#### [Multiple Choice Question]

1. The line  $y = x + 3$  intersects the circle  $x^2 + y^2 = 9$  at two points. The  $x$ -coordinates of these two points are

- A.  $x = 3, x = -1$
- B.  $x = -3, x = 1$
- C.  $x = 0, x = 3$
- D.  $x = 0, x = -3$

#### [Questions requiring working]

2. Solve these equations simultaneously:

a)  $4x - y = 3$  and  $10x + 3y = 2$

d)  $xy = 6$  and  $x + y = 5$

b)  $4x = 17 - y$  and  $x - 3y - 1 = 0$

e)  $x^2 + y^2 = 9$  and  $x + y = 3$

3. Determine the coordinates of the point(s) where the parabola  $y = x^2 - 6x + 8$  meets with the line  $y = -2x + 4$ .

4. Find the coordinates of the points of intersection of the circle  $x^2 + y^2 = 5$  and the straight line  $y - x + 3 = 0$ .
5. Determine the coordinates of the point(s) where the hyperbola  $xy = 15$  meets with the line  $y = 2x - 1$ .
6. Find the co-ordinates of the points of intersection of the line  $y = 3x + 1$  with the parabola  $y = x^2 - 3$ .
7. Find the the points of intersection of the function  $y = x^2 - 4x + 3$  and the straight line  $y = 2x - 5$
8. Two points are A (-2, 3) and B (2, 6).
  - a) Find the distance between the two points.
  - b) Determine the equation of the line passing through the two points.
  - c) The line passing through points A and B intersects with another line  $2y + 4x = 1$ . Find the coordinates of the point of intersection between the two lines.
9. Find the coordinates of the points of intersection of the parabola  $y = x^2 - 6x + 8$  and the straight line  $y = x - 4$
10. Find the coordinates of the points of intersection of the parabola  $y = x^2 + 7x + 9$  and the straight line  $y + x + 3 = 0$ .
11. Graph of  $y = x^2 + 4$  and the straight line  $y = x + K$ , have only 1 point of intersection. Determine the value of  $K$ .
12. Find the point of intersection of the curve  $y = -\frac{2}{x}$  with the line  $y = x - 3$ .

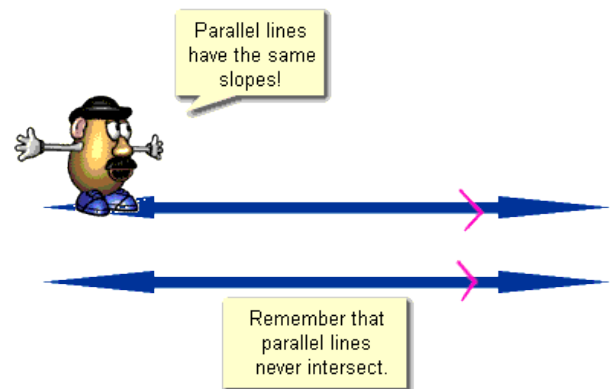
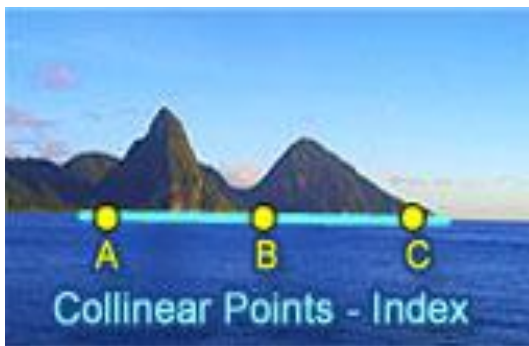
**Lesson of Life:** *Travel by aeroplane would not be possible without the mathematics of airflow and of control systems ~ unknown*

# STRAND FOUR

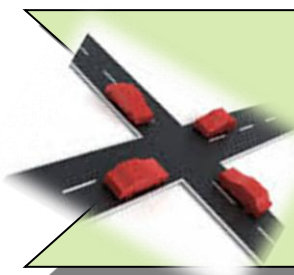
# COORDINATE GEOMETRY

## SUB – STRAND 4.1

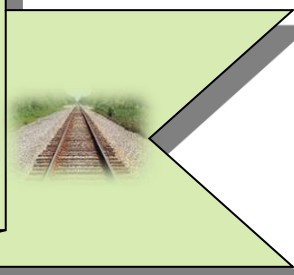
## Applications of Coordinate Geometry



## CONTENT LEARNING OUTCOMES

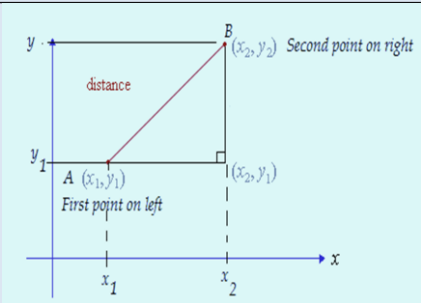
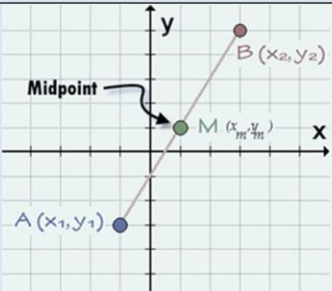
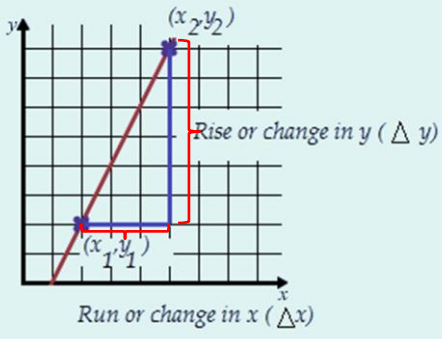


- Gradient/distance/midpoint/ Equation of a line
- Relationship between angle from positive  $x$  – axis and the gradient
- Parallel/Perpendicular lines
- Collinear points



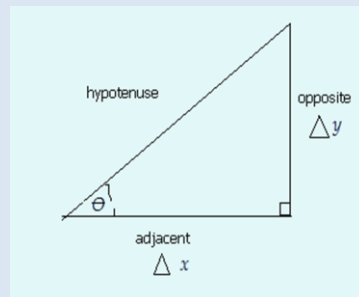
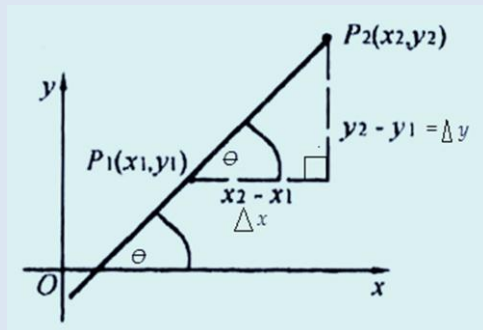
## REVIEW OF FORMULAE

**Note:** There are certain formulae that you have learnt in Years 10 and 11 are given below:

|                                           | Requirements                                                   | Graphically interpretation                                                           | Formula                                                                                                                                                                                                             |
|-------------------------------------------|----------------------------------------------------------------|--------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Distance</b>                           | $(x_1, y_1)$ - first point<br>$(x_2, y_2)$ is the second point |     | $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$                                                                                                                                                                          |
| <b>Midpoint</b>                           | points $(x_1, y_1)$ and $(x_2, y_2)$                           |    | Midpoint: $M(x_m, y_m) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$ ,<br>where $x_m$ is the average of $x$ values and $y_m$ is the average of $y$ values.                                             |
| <b>Gradient or Slope (<math>m</math>)</b> | points $(x_1, y_1)$ and $(x_2, y_2)$                           |  | $m = \frac{\text{Rise}}{\text{Run}} \text{ or } \frac{\Delta y}{\Delta x}$<br>or $\frac{y_2 - y_1}{x_2 - x_1}$<br>$m$ is <b>positive</b><br>$m$ is <b>negative</b><br>$m$ is <b>zero</b><br>$m$ is <b>undefined</b> |
| <b>Equation of line</b>                   | $m$ is gradient and $c$ is the $y$ - intercept.                |                                                                                      | $y = mx + c$                                                                                                                                                                                                        |
|                                           | point $(x_1, y_1)$ with gradient $m$                           |                                                                                      | $y - y_1 = m(x - x_1)$                                                                                                                                                                                              |
|                                           | two points $(x_1, y_1)$ and $(x_2, y_2)$                       |                                                                                      | $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$                                                                                                                                                             |

## GRADIENT OF A LINE MAKING AN ANGLE $\theta$ WITH THE POSITIVE $x$ - AXIS

**Note:** Using Gradient of a line to find the angle  $\theta$  with the positive  $x$  axis: Consider a diagram given below:



Using basic trigonometry, to find angle:

Using TOA:  $\tan \theta = \frac{O}{A}$  Refer to the diagram on the left,

Then:  $\tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$  or  $\frac{\Delta y}{\Delta x}$  Recall this formula is for

gradient, thus

Make angle,  $\theta$ , the subject



$$\tan \theta = m$$

$$\theta = \tan^{-1} m$$

### The two places of an angle:

| Diagram | Angle                                                                                                                     | Example                                                                                                                                                                                                                                                                                        |
|---------|---------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|         | First Quadrant:<br>$\theta = \tan^{-1} m$                                                                                 | Two points (1,1) and (3,2)<br>$(x_1, y_1) \quad (x_2, y_2)$<br>$\tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{3-1} = \frac{1}{2}$<br>$\tan \theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1} (\frac{1}{2}) = 26.27^\circ$                                                           |
|         | Second Quadrant:<br>$\theta = \tan^{-1} m$<br>$\theta_1 = 180 -  \theta $<br>$\theta_1$ is the angle from positive x-axis | Two points (-3,3) and (-1,0)<br>$(x_1, y_1) \quad (x_2, y_2)$<br>$\tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-3}{-1-(-3)} = \frac{-3}{2}$<br>$\tan \theta = (\frac{-3}{2}) \Rightarrow \theta = \tan^{-1} (\frac{-3}{2}) = -56.31^\circ$<br>$\theta_1 = 180 -  -56.31  = 123.69^\circ$ |

**EXAMPLE 1:** What is the gradient of the line  $\frac{x}{4} - \frac{y}{3} = 2$ ?

$$\frac{x}{4} - \frac{y}{3} = 2$$

$$\frac{x}{4} - 2 = \frac{y}{3}$$

Take  $y$  on the right

$$3x \left( \frac{x}{4} - 2 \right) = \frac{y}{3} \times 3$$

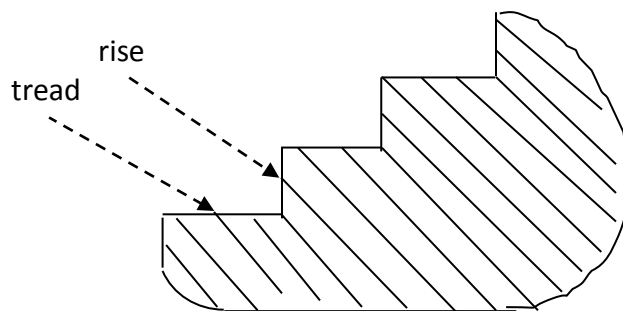
Use distributive law

$$\frac{3}{4}x - 6 = y$$

$m$

$$\text{Thus } m = \frac{3}{4}$$

**EXAMPLE 2:** The diagram given below shows a flight of stairs.



Each step has a 'rise' of 17cm and a 'tread' of 28cm. If a plank was placed on the flight of stairs, calculate the:

- Length of the plank.
- Gradient of the plank.
- Angle the plank would make with the horizontal.

**Answers:**

a) Length

☞ Since it's a right-angle triangle, you may use Pythagoras theorem, i.e.

$$c^2 = a^2 + b^2 \text{ or}$$

$$d(\text{plank})^2 = \Delta x^2 + \Delta y^2$$

$$d(\text{plank})^2 = 17^2 + 28^2$$

$$d(\text{plank}) = \sqrt{1073}$$

$$d(\text{plank}) = 32.76\text{cm}$$

b) Gradient

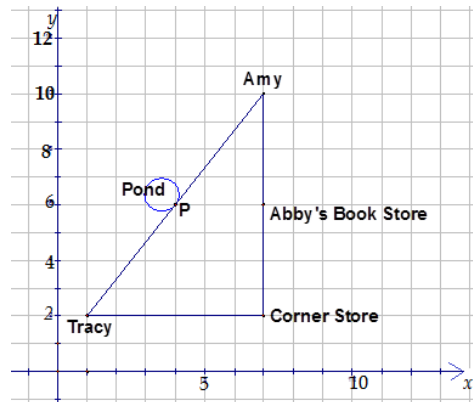
$$m = \frac{\text{Rise}}{\text{Run}} = \frac{17}{28} = 0.61$$

c) Angle

$$m = \tan \theta \Rightarrow \theta = \tan^{-1}(0.61) = 31.26^\circ$$

**EXAMPLE 3:** Tracy wants to visit Amy for her birthday. She decides to walk to the Corner Store way in order to purchase a gift, passing Abby's Book Store to present the gift. On returning home, she will take the shortcut through the park and past the pond.

- a) State the coordinates of the following: Tracy, Amy, Book Store and Pond (point P)
- b) If each unit on the grid represents 200m,
  - i. How many metres will she walk going to Amy's place?
  - ii. How many metres will she walk going home?



- c) As Tracy is walking home through the field, she stops to dangle her feet in the pond that is exactly half way between Amy and Tracy's house.
  - i. Give the coordinates of the pond.
  - ii. What is the distance from the pond to Tracy's home?
- d) Calculate the smallest angle the line joining the points Tracy and Amy makes with the horizontal.

### Answers

- a) Tracy (1, 2)  
Amy (7, 10)  
Book Store (7, 6)  
Pond (4, 6)
- b) Towards Corner store = 6 unit  $\times$  200m = 1200m or 1.2km  
To Amy's place = 8 unit  $\times$  200m = 1600m or 1.6km
  - i. Total distance to Amy's place = 1.2 + 1.6 = 2.8 km
  - ii. **Method 1:** Pythagoras theorem  

$$c^2 = a^2 + b^2$$

$$d(\text{home})^2 = 1.2^2 + 1.6^2$$

$$d(\text{home}) = \sqrt{4}$$

$$d(\text{home}) = 2 \text{ km}$$

**Method 2:** Distance formula Tracy (1, 2) and Amy (7, 10)

$$\begin{aligned}d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\&= \sqrt{(7 - 1)^2 + (10 - 2)^2} \\&= \sqrt{100} \\&= 10 \text{ units} \\&= 10 \times 200 \\&= 2000 \text{ m} \\\therefore 2 \text{ km}\end{aligned}$$

c) Pond is in the middle

i. find the midpoint of the two points: Tracy (1, 2) and Amy (7, 10)

$$\begin{aligned}M(x_m, y_m) &= \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \\&= \left( \frac{1+7}{2}, \frac{2+10}{2} \right) \\&= (4, 6)\end{aligned}$$

ii. Since it's in the middle, then divide by two the distance from Amy to Tracy

$$\text{So } \frac{2 \text{ km}}{2} = 1 \text{ km}$$

d)  $\theta = \tan^{-1} m$ . Recall the gradient formula  $m = \frac{\Delta y}{\Delta x}$

$$\begin{aligned}\theta &= \tan^{-1} \left( \frac{\Delta y}{\Delta x} \right) \\&= \tan^{-1} \left( \frac{1.6}{1.2} \text{ or } \frac{8}{6} \right) \\&= 53.1^\circ\end{aligned}$$

### EXERCISE 38:

#### [Multiple Choice Questions]

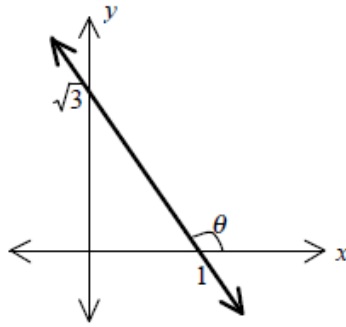
1. The teaching building of a university is mapped on a coordinate grid with the origin being at library. Math's building is located at the point (1, 5) and History's building is located at (4, 9). How far is it from Math's building to History's building?

A.  $\sqrt{13}$       B.  $3\sqrt{2}$       C. 5      D. 7

2. The gradient of the line  $\frac{1}{2}x - 2y = 6$  is

A.  $\frac{-1}{4}$       B.  $\frac{-1}{2}$       C.  $\frac{1}{4}$       D. 1

3. The straight line shown cuts the x – axis at (1,0) and the y – axis at  $(0, \sqrt{3})$ . What is the size of angle to the nearest degree?



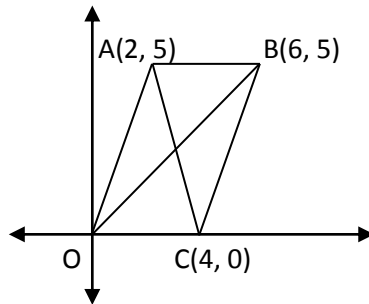
- A.  $\sqrt{3}^\circ$       B.  $60^\circ$       C.  $120^\circ$       D.  $135^\circ$
4. Which one of the following angles does the line joining the points P(1, 5) and Q(5, 7) make with the positive x axis.

- A.  $270.4^\circ$       B.  $233.6^\circ$       C.  $63.4^\circ$       D.  $26.6^\circ$

**[Questions requiring working]**

5. A diameter intersects the circumference of a circle at the points  $(-3,3)$  and  $(1,0)$ . Find the
- length of the diameter
  - length of the radius
  - coordinates of the centre of the circle.
6. Ship A is 8km North and 6km West of a lighthouse. Ship B is 12km South and 4km West of the same lighthouse. What is the shortest distance between the two ships?
7. Prove that the triangle joining the points P(2,7), Q(4,-1) and R(-2,6) is right angled.
8. Show that the points  $(-3,0)$ ,  $(3, 6\sqrt{3})$  and  $(9,0)$  form an equilateral triangle.
9. The lines A(2, 6), B(2, 11) and C(5, p) form an isosceles triangle where  $\overline{AC} = \overline{BC}$ .
- Find the value of p.
  - Determine the coordinates of the midpoint of the line that joins points A and B.
  - Find the equation of a line passing through points A and B.

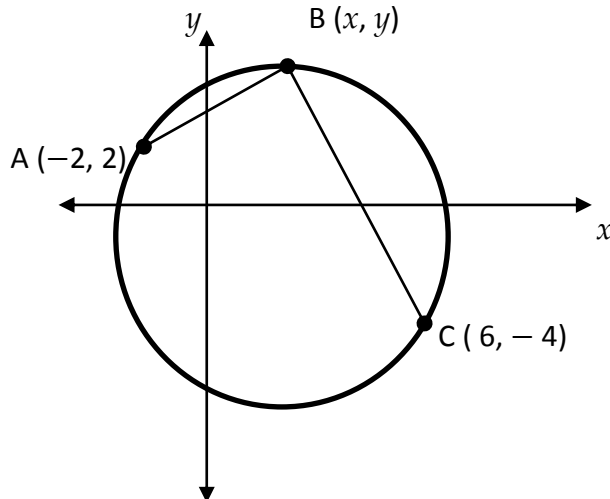
10. Find the equation of a line through the point  $(0, -4)$  which makes an angle of  $135^\circ$  with the positive x-axis.
11. A Rental Company charges \$60 per day plus 30 cents per kilometer for the hire of a particular type of car. Kaviyansh hired one of these cars for one day and travelled  $x$  km.
- Give the equation for the cost, \$ $y$ , that Kaviyansh paid for hiring and for travelling  $x$  km.
  - What do the gradient and  $y$  – intercept in the equation in part (b) represent?
12. The figure OABC shown in the diagram below is a parallelogram. Use it to answer the questions that follow.



- Find the gradient of the line joining points B and C.
  - Find the equation of the line joining B and C.
  - Find the distance of the line joining points O and A.
  - Hence, find the perimeter of the parallelogram.
13. Three points A (2,6), B (2,11) and C (5,p) form an isosceles triangle where  $\overline{AC} = \overline{BC}$ .
- Find the value of p
  - Determine the coordinates of the midpoint of the line that joins points A and B.
14. A line has gradient  $-3$ . Calculate the angle between the line and the positive direction of the x -axis.
15. Seruwaiya and Jocelyn are visiting Fiji. They head off in different directions from the Tourist Information Office in Nadi. An hour later, Seruwaiya has walked 400m East and 100m North while Jocelyn has walked 500m East and 300m South. Calculate the distance between them to the nearest metre.

16. Find the equation of the line which intersects the x axis at 4 and the y axis at -2.

17. Points A  $(-2, 2)$  and C  $(6, -4)$  lie on the circumference of a circle. Further  $\overline{AC}$  is the diameter of the circle. A general point B  $(x, y)$  can lie anywhere on the circle.



- Express the gradients of the chords  $\overline{AB}$  and  $\overline{BC}$  in terms of  $x$  and  $y$ .
- Write down the relation between the two gradients obtained in (i) above.

## PARALLEL LINES

**Note:** Parallel lines have the same slope/gradient. If  $m_1$  is gradient of the first line and  $m_2$  is the slope of the second line, then it is parallel lines if  $m_1 = m_2$

**EXAMPLE 1:** Find the equation of the line parallel to  $3x + 2y - 7 = 0$  and passing through the point  $(2, 4)$ .

**Answers:**

Key words / sentences

- line parallel to  $3x + 2y - 7 = 0$ :

Make  $y$  the subject to find  $m$

$$3x + 2y - 7 = 0$$

$$y = \frac{-3x+7}{2} = \frac{-3x}{2} + \frac{7}{2}$$

$$m_1 = \frac{-3}{2}$$

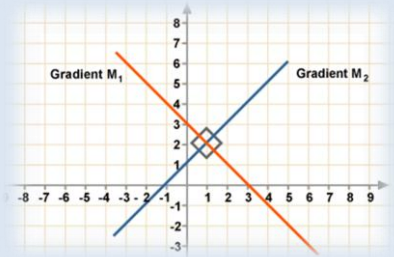
Note that **parallel** means **same gradient**.

$$\text{Thus } m = \frac{-3}{2}$$

- given a point  $(2, 4) \Rightarrow (x_1, y_1)$  and  $m = \frac{-3}{2}$ ,  
Equation of the line is:  $y - y_1 = m(x - x_1)$  Substitute the values provided,  
 $y - 4 = \frac{-3}{2}(x - 2)$
- Make  $y$  the subject  
$$y - 4 = \frac{-3x}{2} + 3$$
$$+ 4 \qquad + 4$$
  
$$\therefore y = \frac{-3x}{2} + 7$$

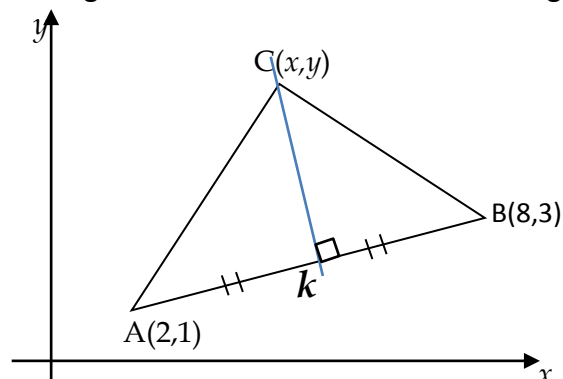
## PERPENDICULAR LINES

**Note:** Recall *Perpendicular* simply means '**at right angle**'. A line is *perpendicular* to another if they meet at **90 degrees**. Graphically,



Moreover, two lines are perpendicular if the **product of gradient** is **negative one** that is  $m_1 m_2 = -1$

**EXAMPLE 2:** The figure ABC below is an isosceles triangle.



- Find the gradient of line segment AB.
- Line  $k$  is a perpendicular bisector of the line segment AB. Determine the coordinates of the midpoint of the line segment AB.
- Write the equation of Line  $k$ .

**Answers:**

a) Gradient formula :  $m = \frac{y_2 - y_1}{x_2 - x_1}$

A (2,1)  $\rightarrow (x_1, y_1)$

and B (8,3)  $\rightarrow (x_2, y_2)$

Substitute the values provided

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{8 - 2} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore m = \frac{1}{3}$$

b) Coordinates of the midpoint

$$M(x_m, y_m) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$M(x_m, y_m) = \left( \frac{2 + 8}{2}, \frac{3 + 1}{2} \right) = \left( \frac{10}{2}, \frac{4}{2} \right)$$

$$\therefore (x_m, y_m) = (5, 2)$$

c) Midpoint will be the first point  $(x_1, y_1) \rightarrow (5, 2)$

- Since perpendicular:

$$m_1 m_2 = -1$$

$$m_2 = \frac{-1}{m_1} = -1 \div \frac{1}{3} = -3$$

- The equation of a line:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -3(x - 5)$$

$$y = -3x + 15 + 2$$

$$\therefore y = -3x + 17$$

## COLLINEAR POINTS

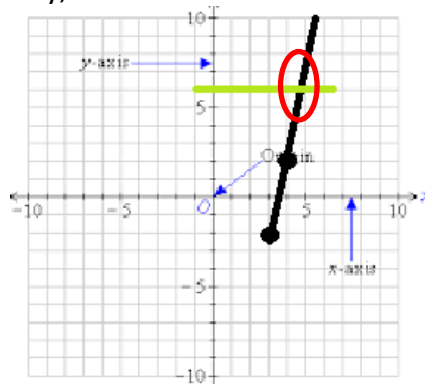
**Note:** Collinear points are points that lie on the same line.



The **gradient** between each pair of points will be the **same**. Consider the points Q, R, S and T. Since they are collinear, their **gradients** will be **same**.

**EXAMPLE 3:** If points P (3,-2), Q (4,2) and R (x,6) are collinear, then find the value of  $x$ .

Graphically,



- Find gradient of  $\overline{PQ}$

$$P(3,-2) \rightarrow (x_1, y_1) \text{ \& } Q(4,2) \rightarrow (x_2, y_2)$$

Use gradient formula:

$$m(\overline{PQ}) = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2 - (-2)}{4 - 3}$$

$$\therefore m(\overline{PQ}) = \frac{4}{1} = 4$$

- **Collinear points** will lie on the same line, means **same gradient**.

$$\therefore m(\overline{PQ}) = m(\overline{QR}) = m(\overline{PR}) = 4$$

- Consider points QR:

$$Q(4,2) \rightarrow (x_1, y_1)$$

$$\text{and } R(x,6) \rightarrow (x_2, y_2)$$

Substitute the given values

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$4 = \frac{6 - 2}{x - 4}$$

- Solve for  $x$ : you may use distributive law

$$4(x - 4) = 4$$

$$4x - 16 = 4$$

$$4x = 20$$

$$x = 5$$

Thus  $x = 5$

### EXERCISE 39:

#### [Multiple Choice Question]

1. The equation of the line containing the point (1,2) and is perpendicular to the line with equation  $2x + 4y = 3$  is

A.  $y = 2x$     B.  $2y = x$     C.  $y = 2x - 4$     D.  $2y = x + 2$

#### [Questions requiring working]

2. The following equations represent four straight lines.

(i)  $3y + 4x = 2$

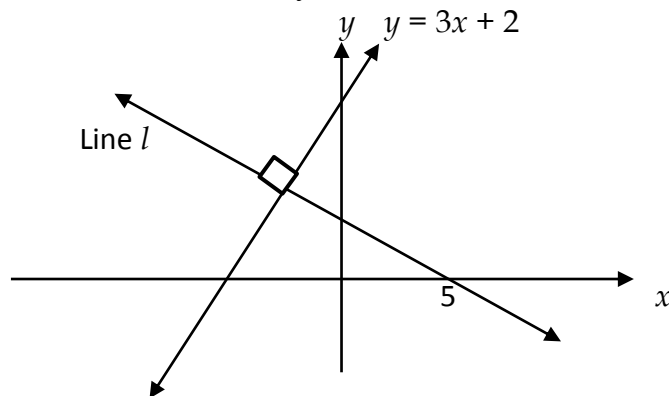
(ii)  $4y = 3x + 2$

(iii)  $3y + 4x = 2$

(iv)  $2 = -4x - 3y$

- a) Find gradient of each line.
  - b) Which two lines are parallel? i.e. have same gradient
  - c) Which two lines are perpendicular?
3. Find the equation of the line through point  $(-2, 3)$  and perpendicular to  $3y = 6 - 2x$ .

4. Test whether (1, 2), (2, 4) and (3, 6) are collinear.
5. Find the equation of the line passing through the origin and perpendicular to  $3y - \frac{2}{3}x = \frac{1}{4}$ .
6. Line  $l$  is perpendicular to the line  $y = 3x + 2$  and meets the  $x$  axis at  $x = 5$ .

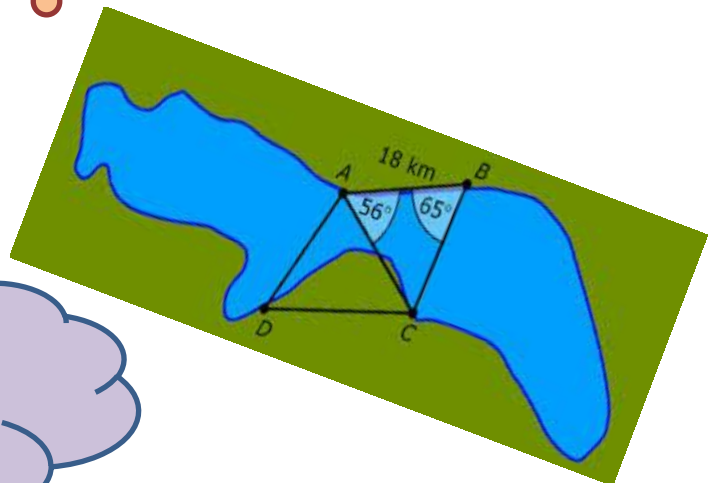
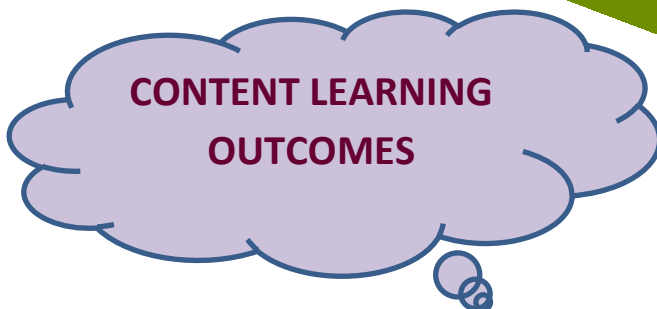
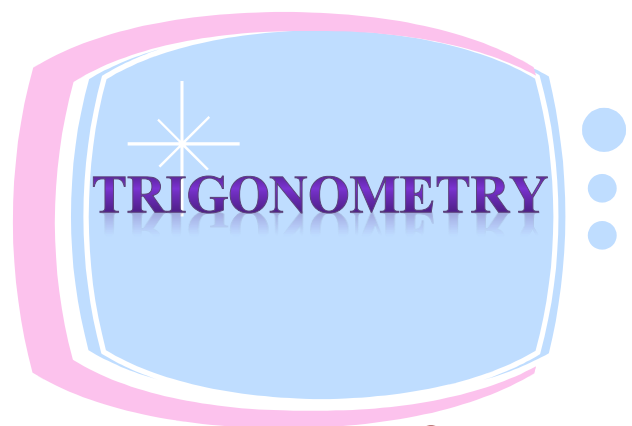


- a) Calculate gradient of each line  $l$ .
  - b) Find the equation of line  $l$ .
7. Test whether the points (1, 2), (3, 9) and (5, 6) are collinear.
  8. For what value of  $p$  will the line  $2x + py = 6$  :
    - a) Parallel to the line  $y = \frac{x}{5} + \frac{4}{5}$
    - b) Perpendicular to the line  $y = -2x + 2$
  9. Three points A, B and C are collinear. If the coordinates of the three points are A (3, 4), B (2, -1) and C (0,  $p$ ), find the value of  $p$ .
  10. The points A ( $x$ , 0), B (0,4) and C (7, 5) form the vertices of a triangle.
    - a) Find the length of the side joining points B and C.
    - b) Given that the gradient of the line joining points A and B is  $-2$ , what is the value of  $x$ ?

Line  $m$  is perpendicular to the line segment AB and passes through the midpoint of AB.

- c) What is the gradient of line  $m$ ?
- d) What is the midpoint of the line segment AB?
- e) Hence, what is the equation of line  $m$ ?

**Lesson of Life:** *Music is the pleasure the human mind experiences from counting without being aware that it is counting.* ~Gottfried Leibniz



Sector  
Segment

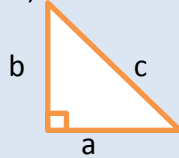
- Pythagoras Theorem and SOH/CAH/TOA
- Sine and Cosine Rule
- Exact values
- Area of Triangle
- Conversion of Angles
- Arc Length, Area of Sector and Segments
- Solving Trigonometric Equations

YOU MAY BE RIGHT, PYTHAGORAS,  
BUT EVERYBODY'S GOING TO LAUGH  
IF YOU CALL IT A "HYPOTENUSE."

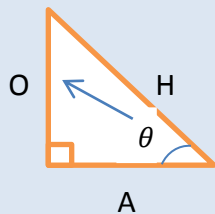
## SINE and COSINE RULE

**Note:** Recall for **right – angle triangles**:

- Use Pythagoras theorem, i.e.  $c^2 = a^2 + b^2$  to solve for the unknown sides where  $c$  is the hypotenuse or the longest length, while 'a' & 'b' are the other 2 lengths of the triangle.



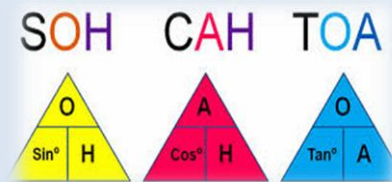
- Use SOH/CAH/TOA to solve for the unknown sides and angles, where O is opposite side of an angle, A is the adjacent side (i.e. next to angle) and H is the hypotenuse or longest side (i.e. opposite of right angle,  $90^\circ$ ).



$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{O}{A}$$



For a **non – right angle triangle**,

Use the **Sine/ Cosine Rule** to find unknown sides and angles.

- SINE RULE

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

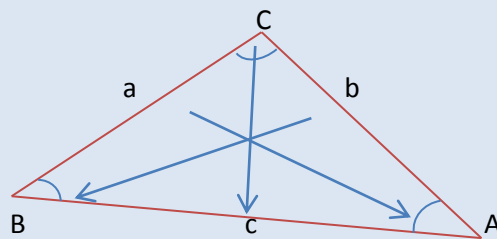
Or :

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

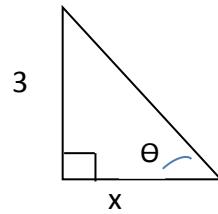
- COSINE RULE

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Where **small letters** [a,b,c] represent the **sides**, and **capital letters** [A,B,C] are representing the opposite **angles**.



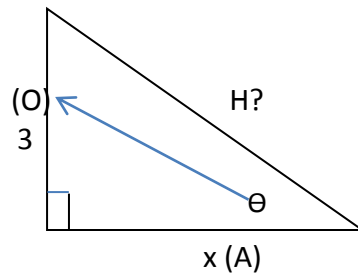
**EXAMPLE 1:** In the right triangle given below, find the value of  $\cos \theta$ ?



**Answer:**

- Aim is to find the value of  $\cos \theta$ , Use 'CAH':

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H} \quad \Rightarrow \text{Label the A and H}$$



- A is marked 'x' and the expression for 'H' is needed in order to find  $\cos \theta$ , thus use Pythagoras theorem

Substitute in the expression:

$$H^2 = a^2 + b^2$$

$$H^2 = x^2 + 3^2$$

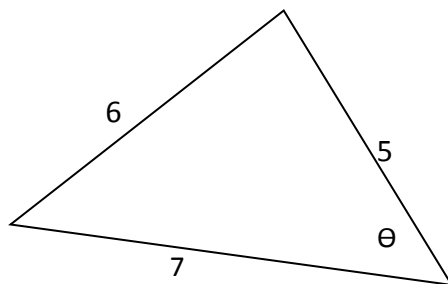
$$H^2 = x^2 + 9$$

$$\sqrt{H^2} = \sqrt{x^2 + 9}$$

$$\therefore H = \sqrt{x^2 + 9}$$

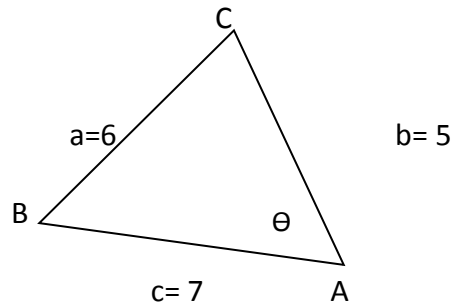
$$\begin{aligned} \cos \theta &= \frac{A}{H} \\ \therefore \cos \theta &= \frac{x}{\sqrt{9 + x^2}} \end{aligned}$$

**EXAMPLE 2:** Find the value for the angle  $\theta$  in the triangle shown below:



**Answer:**

Since only one angle given, let it be A: [Diagram not drawn to scale]



Use Cosine Rule to find the missing angle:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$2bc \cos A = b^2 + c^2 - a^2$  ( take  $2bc \cos A$  to the left and  $a^2$  to the right by doing opposite operations)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{5^2 + 7^2 - 6^2}{2 \times 5 \times 7} = \frac{38}{70}$$

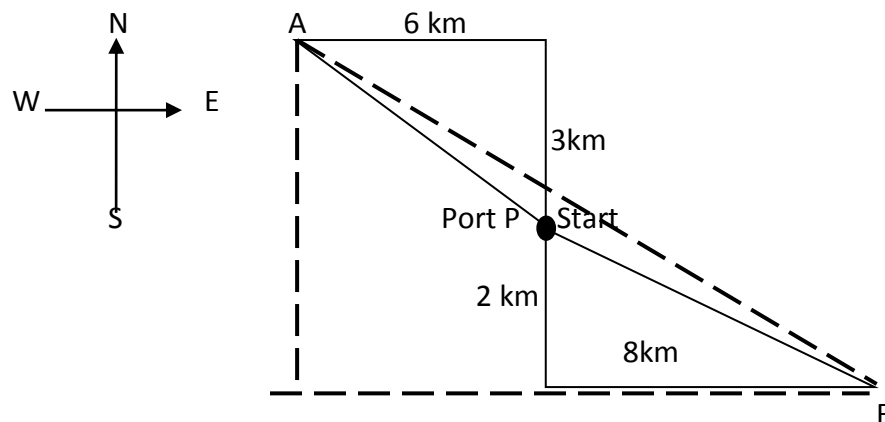
$$\cos A = \frac{38}{70}$$

To solve for A, do the opposite operation:

$$A = \cos^{-1} \left( \frac{38}{70} \right)$$

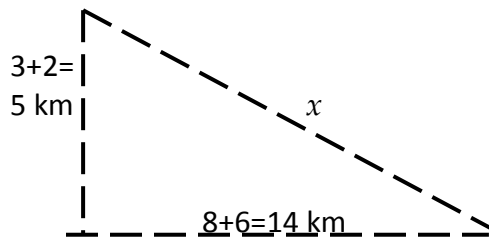
$$A = 57.12^\circ$$

**EXAMPLE 3:** Esekaia sails 3 km North and 6 km West from a Port P to destination A. Samit sails 2 km South and 8 km East from Port P to destination B. How far in a straight line is destination A from B?



**Answer:**

Diagrammatically, we are interested in the bigger triangle,



Using Pythagoras theorem:

$$c^2 = a^2 + b^2$$

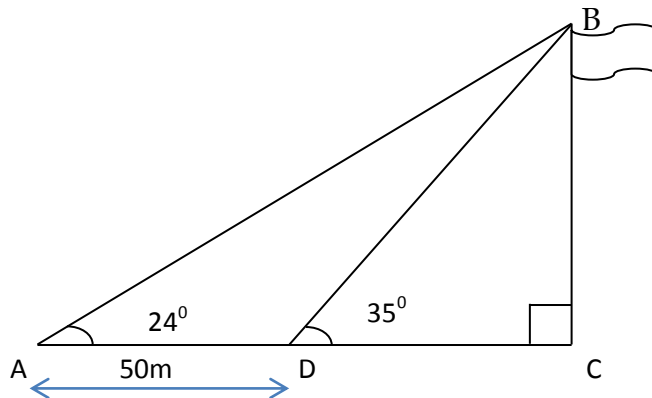
$$x^2 = 5^2 + 14^2$$

$$x^2 = 25 + 196$$

$$x^2 = 221$$

$$\therefore x = \sqrt{221} = 14.87 \text{ km}$$

**EXAMPLE 4:** The diagram below shows 2 angles of elevation of the flagpole,  $24^\circ$  and  $35^\circ$  at points A and D respectively. If point D is 50m from point A, find the distance  $\overline{AB}$ .



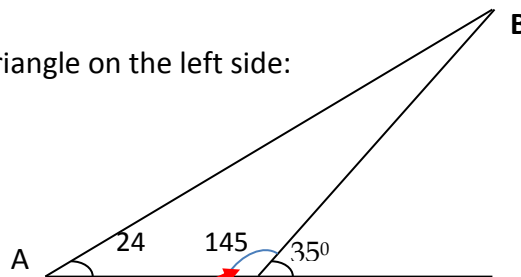
**Answer:**

Consider the non-right angle triangle on the left side:

Using straight lines,

$$180^\circ = 35^\circ + x^\circ$$

$$x = 145^\circ$$



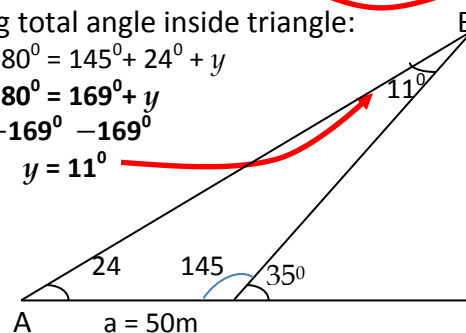
• Using total angle inside triangle:

$$180^\circ = 145^\circ + 24^\circ + y$$

$$180^\circ = 169^\circ + y$$

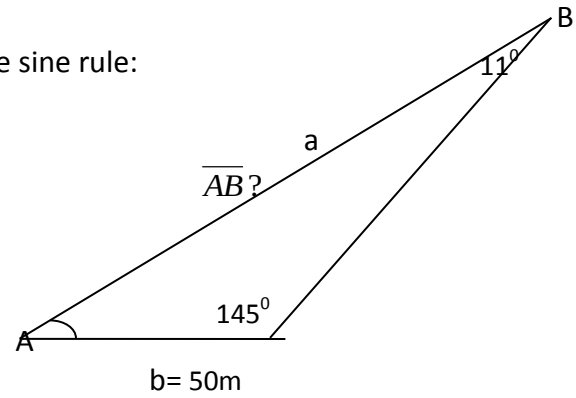
$$-169^\circ \quad -169^\circ$$

$$y = 11^\circ$$



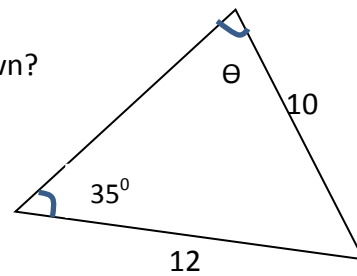
- Since we need to find the length  $\overline{AB}$ , use sine rule:

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{\overline{AB}}{\sin 145} &= \frac{50}{\sin 11} \times \sin 145 \\ \frac{\overline{AB}}{\sin 145} \times \sin 145 &= \frac{50}{\sin 11} \times \sin 145 \\ \overline{AB} &= \frac{50 \sin 145}{\sin 11} \\ \therefore \overline{AB} &= 150.30\text{m}\end{aligned}$$

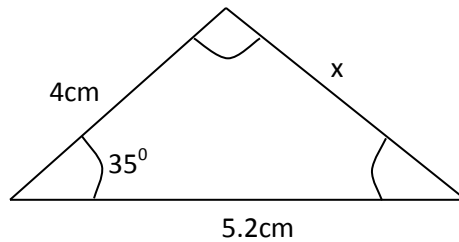


#### EXERCISE 40:

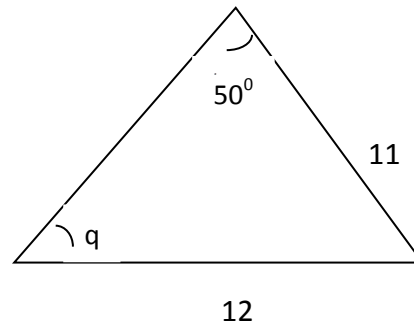
- What is the value of  $\theta$  in the triangle shown?



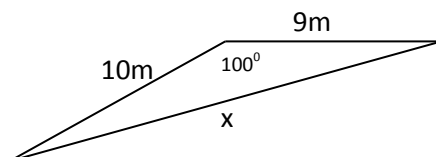
- Find the value of  $x$  in the given triangle.



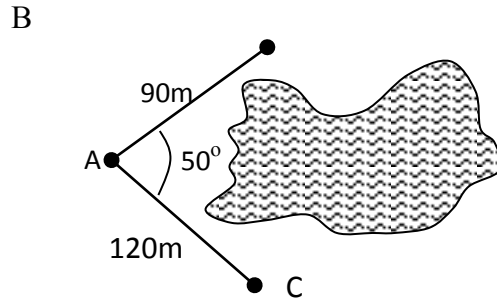
- Find the value of angle marked  $a$ .



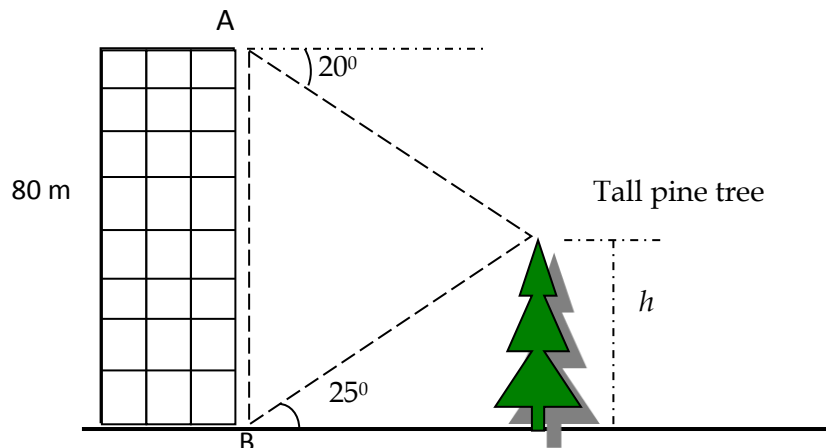
- Use the cosine rule to solve for  $x$ .



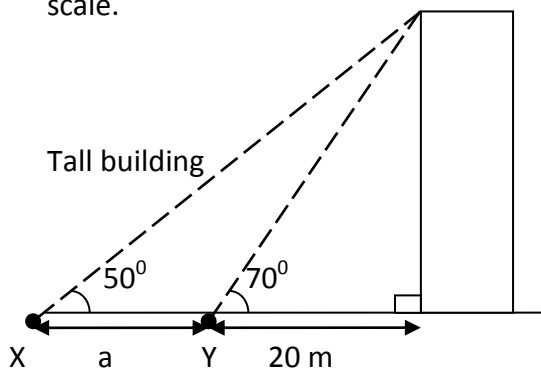
5. The distance between two points B and C on opposite sides of a lake is required. A surveyor locates a point A which is 90m from point B and 120m from point C. Angle BAC is  $50^\circ$ . Calculate the length  $\overline{BC}$ .



6. A private plane flies 1.3 hours at 110 mph on a bearing of  $40^\circ$ . Then it turns and continues another 1.5 hours at the same speed, but on a bearing of  $130^\circ$ . At the end of this time, how far is the plane from its starting point? What is its bearing from that starting point?
7. When the top of a tall pine tree is viewed from the top of a 8 – storey building (point A) 80 m above the ground, the angle of depression =  $20^\circ$  and when it is viewed from point B on the ground, the angle of elevation =  $25^\circ$ . If points A and B are on the same vertical line, find  $h$ , the height of the tall pine tree. (Diagram not to scale)



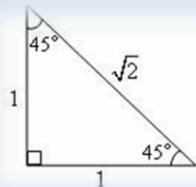
8. Rajjie is stationed at a Point Y, 20 m from the base of a tall building. He looks up to the top of the building at an angle of  $70^\circ$ . Diagram not to scale.



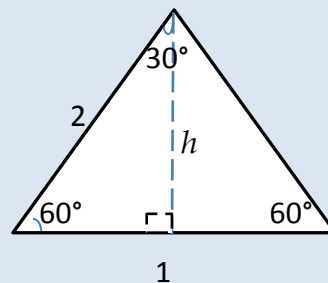
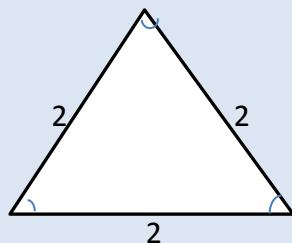
- a) How high is the building?  
b) Rajjie then moves back some distance so that he stands at Point X and now looks to the top of the building at an angle of  $50^\circ$ . Calculate the distance 'a'.

## FINDING EXACT VALUES OF SINE, COSINE AND TANGENT

There are two special triangles in trigonometry. One is the **(45°– 45°– 90°)** isosceles right triangle with both the base and height of length one unit.

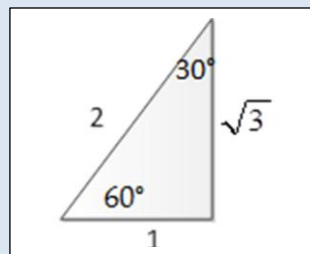


The other is the triangle **30°- 60°- 90°** which is made by dividing an equilateral triangle of length 2 units into two halves.



Using the Pythagorean Theorem, the third side  $h$  yields:

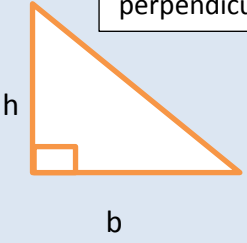
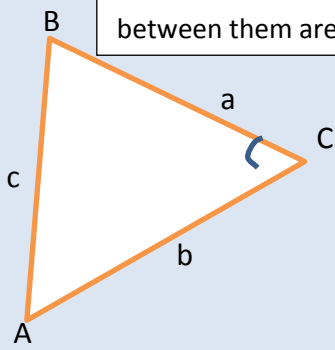
$$\begin{aligned} c^2 &= a^2 + b^2 \\ 2^2 &= h^2 + 1^2 \\ \therefore h^2 &= 2^2 - 1^2 \\ h^2 &= 3 \\ \sqrt{h^2} &= \sqrt{3} \\ h &= \sqrt{3} \end{aligned}$$



The table below shows the ratios of sides in a surd form:

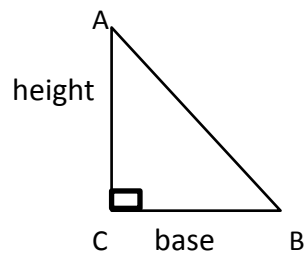
| Trig Function / Angles | 30°                  | 45°                  | 60°                  | 90°       |
|------------------------|----------------------|----------------------|----------------------|-----------|
| Sin $\theta$           | $\frac{1}{2}$        | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1         |
| Cos $\theta$           | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$        | 0         |
| Tan $\theta$           | $\frac{1}{\sqrt{3}}$ | 1                    | $\sqrt{3}$           | undefined |

## AREA OF TRIANGLE

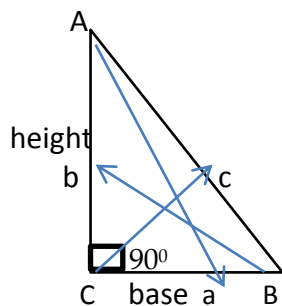
| Right - angled triangle                                                                                                        | Non - right angled triangle                                                                                                                                                                                                                                 |
|--------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p style="border: 1px solid black; padding: 5px; display: inline-block;">Given Base and perpendicular Height</p>               | <p style="border: 1px solid black; padding: 5px; display: inline-block;">Two sides and the angle between them are known</p>                                                                                                                                 |
|                                               |                                                                                                                                                                           |
| <p>Area: <span style="border: 1px solid black; padding: 5px; display: inline-block;"><math>A = \frac{1}{2}bh</math></span></p> | <p><span style="border: 1px solid black; padding: 5px; display: inline-block;"><math>A = \frac{1}{2}ab \sin C</math></span> or</p>                                                                                                                          |
| <p>Where b – base and<br/>h – Perpendicular height</p>                                                                         | <p><span style="border: 1px solid black; padding: 5px; display: inline-block;"><math>A = \frac{1}{2}ac \sin B</math></span> or <span style="border: 1px solid black; padding: 5px; display: inline-block;"><math>A = \frac{1}{2}bc \sin A</math></span></p> |

**Example 1:** Show that the  $A = \frac{1}{2} ab \sin C$  is equivalent to  $\frac{1}{2} \text{base} \times \text{height}$ .

Consider a right angle triangle.



- Label the sides and angles:



- Use the formula for non – right angle triangle:

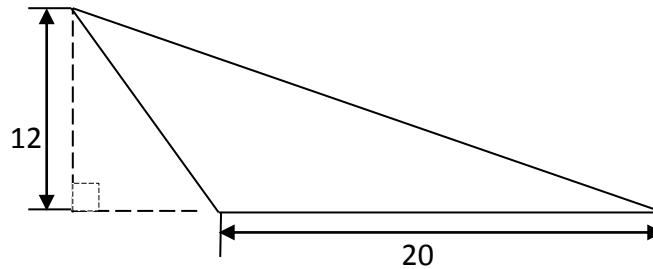
$$\text{Area } \Delta = \frac{1}{2} a \bullet b \sin C$$

$$= \frac{1}{2} \text{base} \times \text{height} (\sin 90^\circ)$$

$$\therefore \text{Area } \Delta = \frac{1}{2} \text{base} \times \text{height}$$

☛ Since  $90^\circ$  is one

**Example 2:** What is the area of the triangle given below?



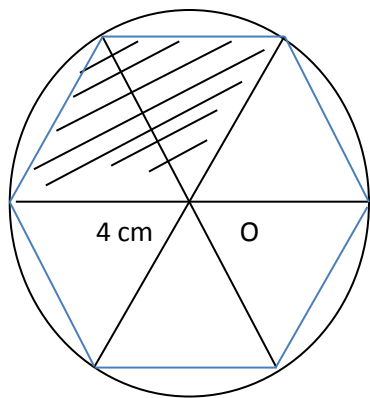
**Answer:**

Perpendicular Height:  $h = 12$

Base:  $b = 20$

Thus Area =  $\frac{1}{2}bh = \frac{1}{2} \times 20 \times 12 = 120 \text{ cm}^2$

**EXAMPLE 2:** The diagram below shows a regular hexagon inscribed in a circle of radius 4 cm at centre O. (Diagram not to scale).



Calculate:

- Angle of each sector formed.
- the area of one of the triangle
- the area of shaded region.

a) Angle  
region

$$T_{angle} = 360^\circ$$

$$6x = 360^\circ$$

$$x = 60^\circ$$

b) area of one of the triangle

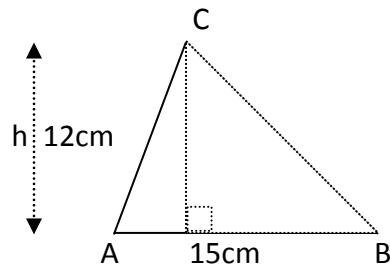
$$\begin{aligned} A_{\Delta} &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}(4 \times 4)\sin 60^\circ \\ &= 4\sqrt{3}\text{cm}^2 \end{aligned}$$

c) area of shaded

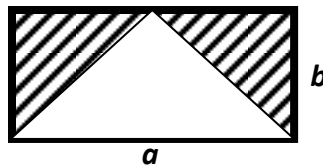
$$\begin{aligned} A &= 2 \times A_{\Delta} \\ &= 2 \times 4\sqrt{3} \\ &= 8\sqrt{3}\text{cm}^2 \end{aligned}$$

### EXERCISE 41:

1. Calculate the area of triangle ABC shown below.

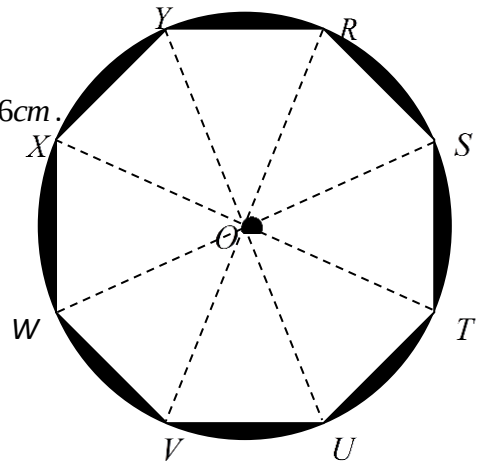


2. Use the diagram given to answer the questions that follow.

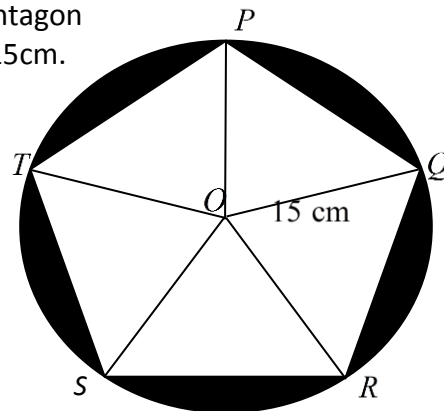


- a) Write an expression which could be used to calculate the area of the shaded region.
- b) If  $a = 120$  cm and  $b = 40$  cm, calculate the area of the shaded region.

3. The diagram below shows a regular octagon  $RSTUVWXY$  inscribed in a circle of radius  $16$  cm. That is  $\overline{OR} = 16$  cm.



- a) Find the area of triangle  $ROS$ .
  - b) Find the area of the octagon.
4. The diagram below shows a regular pentagon  $PQRST$  inscribed in a circle of radius  $15$  cm.



- a) Show that angle  $POQ$  is  $72^\circ$ .
- b) Find the area of triangle  $POQ$ .
- c) Find the area of pentagon.

**Lesson of Life:** Go down deep enough into anything and you will find mathematics ~ Dean Schlicter

## CONVERSION OF DEGREES TO RADIANS and VICE VERSA

**Another unit for measuring angles is radians.** Why learn radians? -

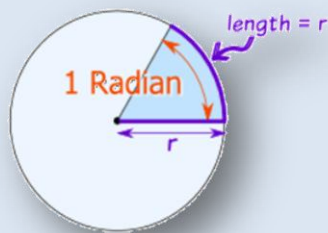
What's wrong with degrees?

It is used in many areas of mathematics. In science and engineering, **radians** are much more convenient (and common) than degrees. Many problems can be solved directly if we know the angle in radians especially in **calculus** (differentiation and integration).

**How do we measure angles?**

There are two main units of measuring angles: degrees (symbol  $^{\circ}$ ) and radians.

**One Radian:** the angle made when we take the radius and wrap it along the edge of a circle.



**One complete circle =  $360^{\circ}$  or  $2\pi$  radians**

- From above,

$$\begin{aligned}\pi \text{ radians} &= 180^{\circ} \\ 2\pi \text{ radians} &= 360^{\circ}\end{aligned}$$

- To convert degrees to radians: multiply by  $\pi/180$

$$\frac{\theta^{\circ}}{180^{\circ}} \times \pi = \text{angle in radians}$$

- To convert radians to degrees: divide by  $\pi/180$  or multiply by  $180/\pi$

$$\text{Angle in radians} \times \frac{180^{\circ}}{\pi} = \theta^{\circ}$$

**Conversion of some common angles**

The table shows the conversion of some **common angles**.

| Units   | Values          |                 |                 |                 |                  |                  |               |                  |                  |                  |               |
|---------|-----------------|-----------------|-----------------|-----------------|------------------|------------------|---------------|------------------|------------------|------------------|---------------|
| Radians | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\pi$         | $\frac{5\pi}{4}$ | $\frac{3\pi}{2}$ | $\frac{7\pi}{4}$ | $2\pi$        |
| Degrees | $30^{\circ}$    | $45^{\circ}$    | $60^{\circ}$    | $90^{\circ}$    | $120^{\circ}$    | $135^{\circ}$    | $180^{\circ}$ | $225^{\circ}$    | $270^{\circ}$    | $315^{\circ}$    | $360^{\circ}$ |

**Example 1:** Convert the following angles to radians

a)  $45^\circ$

$$\begin{aligned}\text{angle} &= \frac{\theta^\circ}{180^\circ} \times \pi \\ &= \frac{45^\circ}{180^\circ} \times \pi \\ &= \frac{\pi}{4}\end{aligned}$$

b)  $15^\circ$

$$\begin{aligned}\text{angle} &= \frac{\theta^\circ}{180^\circ} \times \pi \\ &= \frac{15^\circ}{180^\circ} \times \pi \\ &= \frac{\pi}{12}\end{aligned}$$

c)  $25^\circ$

$$\begin{aligned}\text{angle} &= \frac{\theta^\circ}{180^\circ} \times \pi \\ &= \frac{25^\circ}{180^\circ} \times \pi \\ &= \frac{5\pi}{36}\end{aligned}$$

**Example 2:** Convert the following angles to degrees

a)  $\frac{\pi}{5}$

$$\begin{aligned}\text{angle} &= \theta \times \frac{180^\circ}{\pi} \\ &= \frac{\pi}{5} \times \frac{180^\circ}{\pi} \\ &= 36^\circ\end{aligned}$$

b)  $\frac{5\pi}{2}$

$$\begin{aligned}\text{angle} &= \theta \times \frac{180^\circ}{\pi} \\ &= \frac{5\pi}{2} \times \frac{180^\circ}{\pi} \\ &= 450^\circ\end{aligned}$$

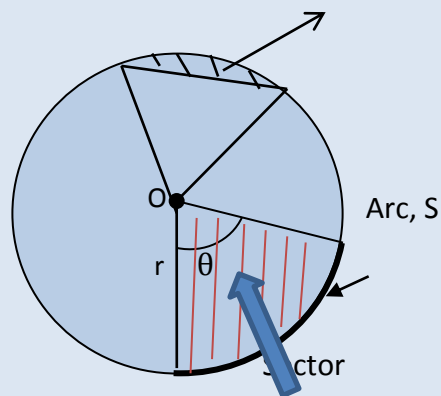
c)  $\frac{4\pi}{7}$

$$\begin{aligned}\text{angle} &= \theta \times \frac{180^\circ}{\pi} \\ &= \frac{4\pi}{7} \times \frac{180^\circ}{\pi} \\ &= \frac{720^\circ}{7} \\ &\text{or } 102.86^\circ\end{aligned}$$

## ARC LENGTH, AREA OF SECTOR and SEGMENT

**Note:** Refer to the Parts of a circle

Segment



where  $r$  is the radius of the circle  
 $\theta$  is the angle in **radians**  
 $O$  as **center** of circle

### 1. ARC LENGTH (S)

The arc length is the measure of the distance along the curved line making up the arc.

**Formulae:**

$$S = r \theta$$

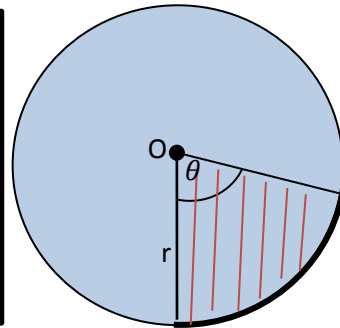
[Angle must be in radians]

### 2. AREA OF SECTOR

Sector is the area enclosed by an arc and the two radii

**Formulae:**

$$\begin{aligned}\text{Area}_{\text{sector}} &= \frac{\theta}{360^\circ} \times \pi r^2 \\ \text{Since } 360^\circ &= 2\pi \text{ radians, substituting yields} \\ \text{Area}_{\text{sector}} &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \frac{\theta}{2} \times r^2, \text{ rearranging} \\ &= \frac{1}{2} r^2 \theta \\ &\text{[angle must be in radians]}\end{aligned}$$

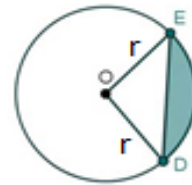


### 3. AREA OF SEGMENT

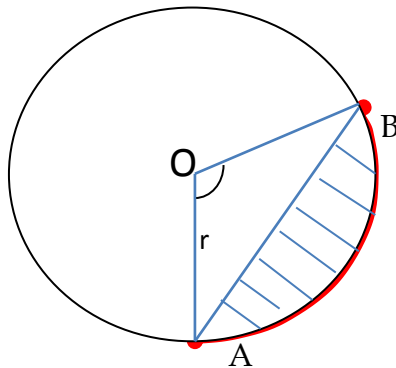
Segment is the region bounded by a chord and an arc.

**Formulae:**

$$\begin{aligned}\text{Area}_{\text{segment}} &= \text{Area}_{\text{sector}} - \text{Area}_{\text{triangle}} \\ &= \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin \theta \\ &\text{or} \\ &= \frac{1}{2} r^2 (\theta - \sin \theta) \\ &\text{[angle must be in radians]}\end{aligned}$$



**Example 3:** The diagram below shows a circle of radius  $r$ .  $OAB$  is a sector of the circle and has an angle of  $95^\circ$ .



- Convert the angle to radians.
- Given that the segment has an area of  $100 \text{ cm}^2$ , calculate radius of the circle.
- Calculate the length of minor arc AB.
- Hence or otherwise, determine the perimeter of the sector.

**Answers:**

- i. Conversion to radians

$$\begin{aligned}\text{angle} &= \frac{\theta^\circ}{180^\circ} \times \pi \\ &= \frac{95^\circ}{180^\circ} \times \pi \\ &= \frac{19\pi}{36}\end{aligned}$$

- iii. Length of minor arc AB  
add all sides

$$\begin{aligned}S &= r\theta \\ &= 17.83 \times \frac{19\pi}{36} \\ &= 28.82 \text{ cm}\end{aligned}$$

- ii. radius of the circle

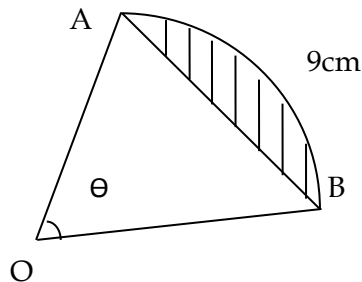
$$\begin{aligned}\text{Area}_{\text{segment}} &= \frac{1}{2} r^2 (\theta - \sin \theta) \\ 100 &= \frac{1}{2} r^2 \left( \frac{19\pi}{36} - \sin \frac{19\pi}{36} \right) \\ 100 &= 0.330934045 r^2 \\ \frac{100}{0.330934045} &= r^2 \\ r^2 &= 302.175 \\ r &= 17.38 \text{ cm}\end{aligned}$$

- iv. Perimeter of the sector:

$$\begin{aligned}P &= \overline{OA} + \overline{OB} + \text{Length of arc AB} \\ &= 17.38 + 17.38 + 28.82 \\ &= 63.58 \text{ cm}\end{aligned}$$

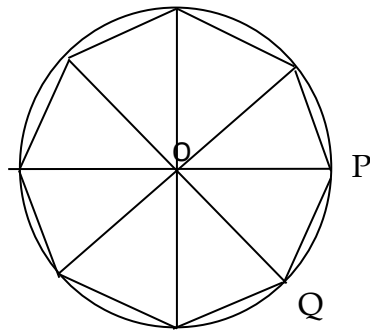
**EXERCISE 42:**

- The diagram below shows a sector of a circle with the radius of 6cm and the length of the arc is 9cm.



- Show that the angle  $\theta = 1.5$  rad.
- Calculate the area of the sector OAB.
- Calculate the area of the shaded segment.

2. The diagram below shows a regular octagon inscribed in a circle of radius 4 cm with centre O.

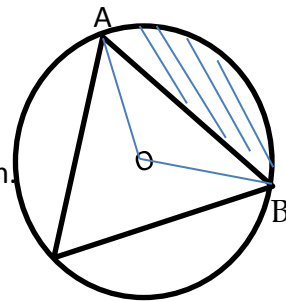


Calculate the following:

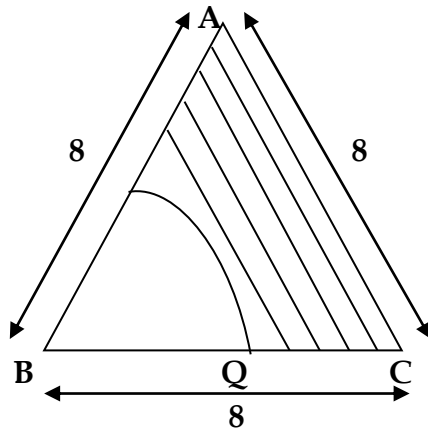
- Length of arc PQ
- Area of sector OPQ
- Area of triangle OPQ
- Area of the octagon

3. An equilateral triangle is inscribed in a circle with centre O. The radius of the circle is 5 cm.

- Find the area of the sector OAPB.  
P
- Find the area of the  $\triangle AOB$
- Hence, calculate the area of the shaded region.



4.



In the diagram given above, P is the midpoint of AB; Q is the midpoint of BC

- Find the length of the arc PQ.
- Determine the area of the sector PBQ to the nearest whole number
- Find the area of the shaded region.

**Lesson of Life:** *The physical sciences (chemistry, physics, oceanography, astronomy) require mathematics for the development of their theories - anonymous*

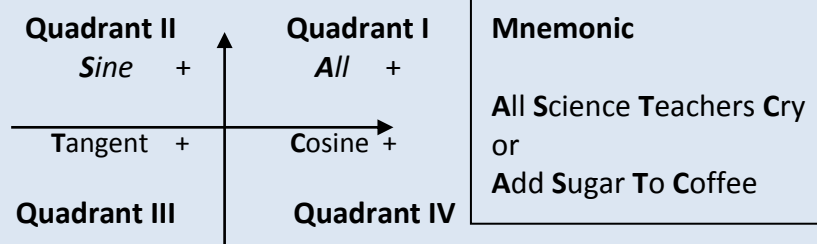
## SOLVING TRIGONOMETRIC EQUATIONS

### Note:

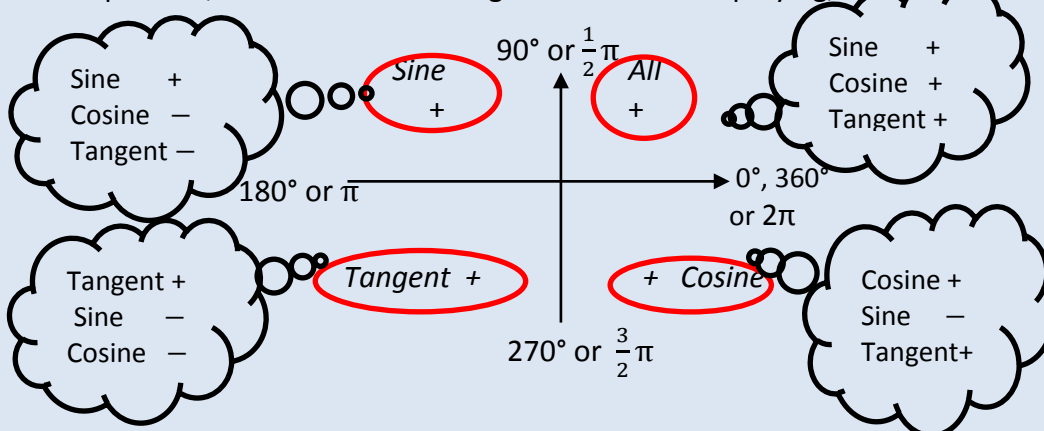
When solving any trigonometric equation, emphasis must be given to the angle,  $\theta$ , which can be either in degrees or radians.

To solve for  $\theta$ , follow solving an algebraic equation:

- The value consisting  $\theta$ , to be removed last.
- Do opposite operation on both sides of the equation till you reach the trigonometric expression containing sine, cosines or tangent.
- At this point in time, keep in mind that there will be at least two angles, within  $0^\circ$  to  $360^\circ$  or  $2\pi$  radians.
- If Trig expression is positive, then you will directly get the acute angle  $\theta_1$ . If Trig expression is negative, then You may (ignore the negative sign to) get the acute angle  $\alpha$ , use this to find the angle  $\theta_1$
- Use quadrants to find the other angle  $\theta_2$ . Angles will be considered from the positive x – axis.



- If you look at the quadrants, the designated trig expressions will be positive, the others will be negative. Further simplifying,



**Example 1:** Solve  $\tan \theta - 1 = 0, 0 \leq \theta \leq 2\pi$

Last

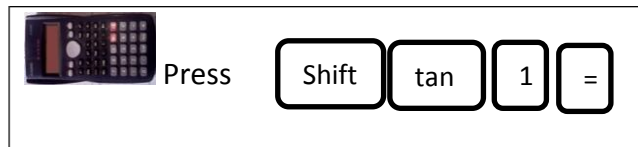
$\tan \theta - 1 = 0, 0 \leq \theta \leq 2\pi$  means that angle to be between  $0 - 2\pi$

$$\tan \theta - 1 + 1 = 0 + 1,$$

$$\tan \theta = 1$$

$$\theta_1 = \tan^{-1} 1$$

$$\theta_1 = 45^\circ$$

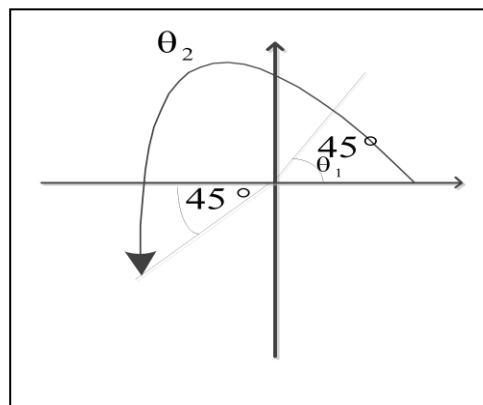


We reached at the trig expression: Consider the two quadrants. Find the acute angle in Q I. Note that calculator Mode to be in degrees.

- Use quadrants to find the angle  $\theta_2$ . Consider sign (+) of 'tan', that is in Q III

$$\begin{aligned}\theta_2 &= 180 + \theta_1 \\ &= 180 + 45 \\ &= 225^\circ\end{aligned}$$

$$\begin{aligned}\theta &= 45^\circ, 225^\circ \text{ or} \\ \theta &\in \{45^\circ, 225^\circ\}\end{aligned}$$



**Example 2:** Find the solution set for  $2\cos \theta + \sqrt{3} = 0, 0^\circ \leq \theta \leq 360^\circ$

$2\cos \theta + \sqrt{3} = 0, 0^\circ \leq \theta \leq 360^\circ$  Means that angle to be between  $0 - 360^\circ$

Last

$$2\cos \theta + \sqrt{3} - \sqrt{3} = 0 - \sqrt{3}$$

$$\frac{2\cos \theta}{2} = \frac{-\sqrt{3}}{2}$$

$$\cos \theta = \frac{-\sqrt{3}}{2}$$

We reached at the trig expression: Consider the two quadrants. But before that, find the acute angle by ignoring the negative sign (-). Note that calculator Mode to be in degrees.

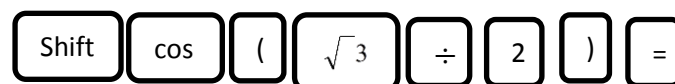
- **Acute angle:**

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$$

$$\alpha = 30^\circ$$



When dealing with surds,  
press the division sign ( $\div$ ), that is Press

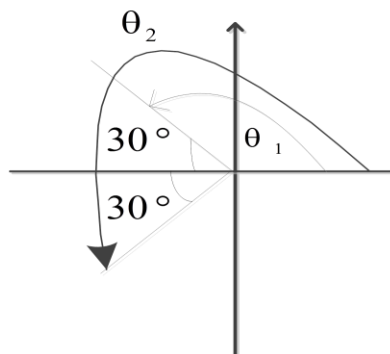


- Use quadrants to find the angles  $\theta_1$  and  $\theta_2$ . Consider negative sign ( $-$ ) of Cos, that is in Q II / III

$$\theta_1 = 180 - 30 = 150^\circ$$

$$\theta_2 = 180 + 30 = 210^\circ$$

$$\theta = 150^\circ, 210^\circ \text{ or } \theta \in \{150^\circ, 210^\circ\}$$



**EXAMPLE 3:** Solve the trigonometric equation  $\sin(x + 30^\circ) = 0.4$ , where  $-180^\circ \leq x \leq 180^\circ$ .

**Acute angle:**

$$\sin(x + 30^\circ) = 0.4$$

$$(x + 30^\circ) = \sin^{-1} 0.4$$

$$\alpha = 23.58^\circ$$



Press

It already has trig expression:

Consider the two quadrants. Note that  
calculator Mode to be in degrees.



- Use quadrants to find the angles  $\theta_1$  and  $\theta_2$ . Consider positive sign ( $+$ ) of sin, that is in Q I / II

$$\theta_2 = 180 - \alpha = 180 - 23.58 = 156.42^\circ \quad \alpha = \theta_1 = 23.58^\circ$$

**QI & II :**

$$QI : x + 30^\circ = 23.58,$$

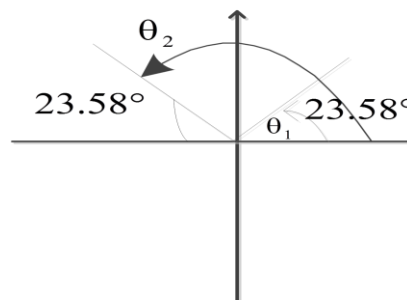
$$\textcircled{x} + 30^\circ = 23.58,$$

$$x = -6.42^\circ$$

$$QII : \textcircled{x} + 30^\circ = 156.42$$

$$x = 126.42^\circ$$

$$\therefore x \in \{-6.42^\circ, 126.42^\circ\}$$



So far the coefficient of the variable was one. If the coefficient changes to other number, let's then expect the solution to double, that is

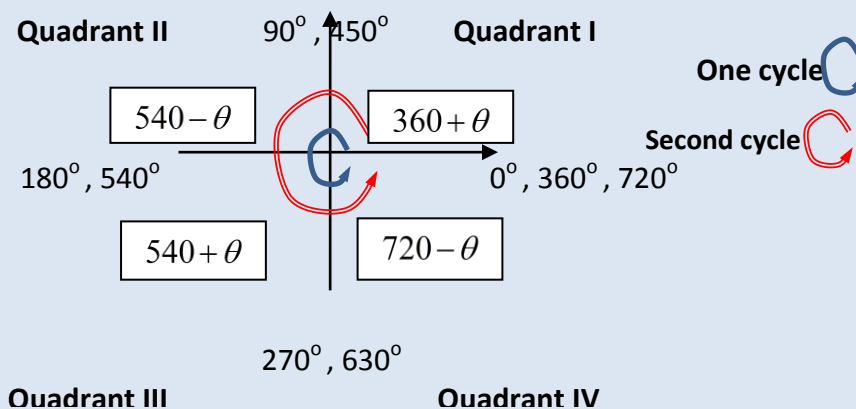
$1x \Rightarrow$  two solutions

$2x \Rightarrow$  four solutions

$3x \Rightarrow$  six solutions and so on.

For the first cycle, angle is found the normal way as before.

For the second cycle, the angles can be found as:



**EXAMPLE 4:** Solve  $2 \cos 2x = \sqrt{3}$  for  $0 \leq x \leq 2\pi$

$$2 \cos 2x = \sqrt{3}$$

Last

$0 \leq x \leq 2\pi$  Angle to be between  $0 - 360^\circ$

$$\frac{2 \cos 2x}{2} = \frac{\sqrt{3}}{2}$$

$$\cos 2x = \frac{\sqrt{3}}{2}$$

We reached at the trig expression and the value in front of the variable is 2, so it has 4 solutions. If you find easier in degrees, you may do so. Once all solutions are found, then you can convert to radians as the question requires you to give answers in radians.

**Acute angle:**

$$2x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$$

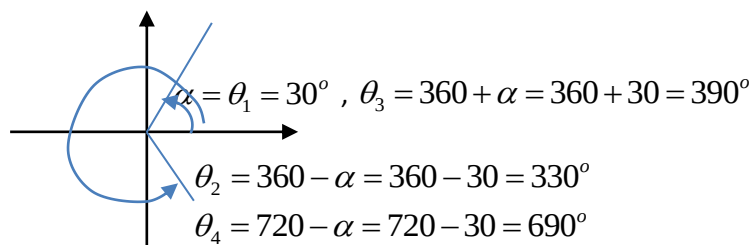
$$\alpha = 30^\circ$$



(when dealing with surds, press the division sign ( $\div$ ), that is Press

$$\text{Shi} \quad \text{co} \quad ( \quad \sqrt{3} \quad \div \quad 2 \quad ) \quad =$$

- Use quadrants to find the **four** angles: Cos is positive in Quadrants I / IV



$$2x = \{30^\circ, 330^\circ, 390^\circ, 690^\circ\}$$

Divide by 2, to all the angles

$$\frac{2x}{2} = \left\{ \frac{30^\circ}{2}, \frac{330^\circ}{2}, \frac{390^\circ}{2}, \frac{690^\circ}{2} \right\}$$

$$x = \{15^\circ, 165^\circ, 195^\circ, 345^\circ\}$$

To convert in radians, divide by 180 and multiply by  $\pi$  :

$$x = \left\{ \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12} \right\}$$

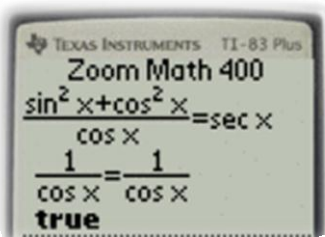
#### EXERCISE 43: Solve

- $\sin\left(x - \frac{\pi}{4}\right) = 0$ , for  $0 \leq x \leq 2\pi$ .
- $2 \sin x + \sqrt{3} = 0$ , for  $0^\circ \leq x \leq 360^\circ$ .
- $\cos 2x = \frac{1}{2}$ , for  $0^\circ \leq x \leq 360^\circ$ .
- $2 \cos\left(\theta - \frac{\pi}{4}\right) = 1$  for  $0 \leq \theta \leq 2\pi$ .
- $\tan 2x - 1 = 0$ , for  $0^\circ \leq x \leq 360^\circ$
- $\sin\left(2x + \frac{\pi}{4}\right)$  for  $0 \leq x \leq 2\pi$ .

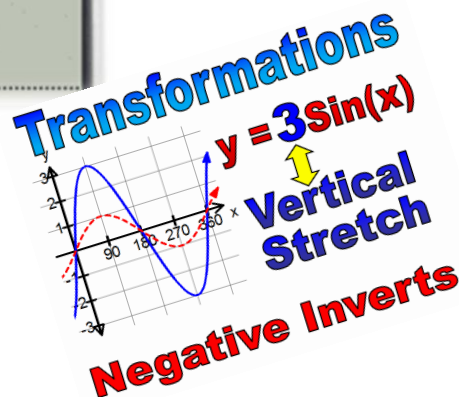
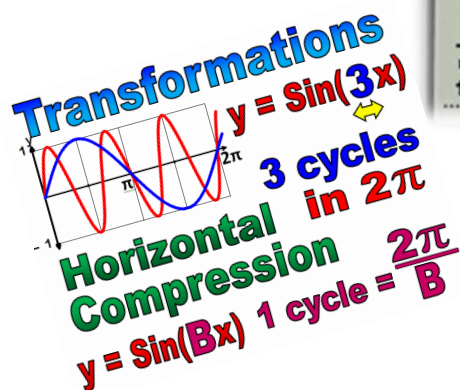
**Lesson of Life:** *If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.* ~John Louis von Neumann

## SUB – STRAND 5.2

## Trigonometric Identities and Graphs



TEXAS INSTRUMENTS TI-83 Plus  
Zoom Math 400  
 $\frac{\sin^2 x + \cos^2 x}{\cos x} = \sec x$   
 $\frac{1}{\cos x} = \frac{1}{\cos x}$   
true



### CONTENT LEARNING OUTCOMES

- Basic trigonometric identities
- Derive from basic identities
- Transform trigonometric graphs

## TRIGONOMETRIC IDENTITIES

**Note:** Some basic formulae:

❶  $\sin^2 \theta + \cos^2 \theta = 1$

❷  $1 + \tan^2 \theta = \sec^2 \theta$

❸  $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$

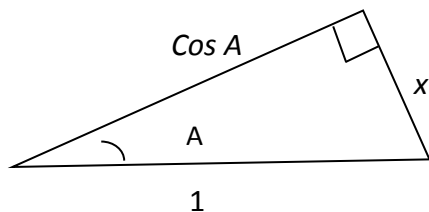
❹  $\tan \theta = \frac{\sin \theta}{\cos \theta}$

❺  $\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$

❻  $\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}}$

❼  $\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} = \frac{\text{adjacent}}{\text{opposite}}$

**EXAMPLE 1:** In the right triangle given below the sides are marked  $\cos A$ , 1 and  $x$ .



Find the expression that represents the side marked  $x$ :

Recall Pythagoras theorem:

$$a^2 + b^2 = c^2 ;$$

$$x^2 + (\cos A)^2 = 1^2 \quad \text{Make } x \text{ the subject}$$

$$x^2 = 1^2 - (\cos A)^2,$$

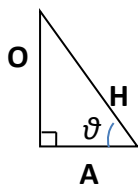
$$x = \sqrt{1 - (\cos A)^2} \quad \text{use identity ❶}$$

$$= \sqrt{(\sin A)^2} = \sin A$$

**EXAMPLE 2:** Proof that  $\sin^2 \vartheta + \cos^2 \vartheta = 1$

**Answer**

- For any right angle triangle, use the expression SOH / CAH



- Start with the **left** side:

$$\sin^2 \vartheta + \cos^2 \vartheta$$

Substitute the expression

$$= \left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2$$

$$= \frac{O^2}{H^2} + \frac{A^2}{H^2}$$

same denominator so collect the numerator

$$= \frac{O^2 + A^2}{H^2}$$

recall Pythagoras theorem

$$\boxed{\begin{array}{l} c^2 = a^2 + b^2 \\ H^2 = O^2 + A^2 \end{array}}$$

$$= \frac{H^2}{H^2}$$

$$= 1$$

Thus  $\sin^2 \vartheta + \cos^2 \vartheta = 1$

**EXAMPLE 3:** Using the identity  $\sin^2 \theta + \cos^2 \theta = 1$  to show that  $1 + \cot^2 \theta = \csc^2 \theta$

Divide the whole equation by  $\sin^2 \theta$ :

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

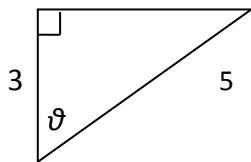
$$1 + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \therefore \cot^2 \theta = \frac{\cos^2 \theta}{\sin^2 \theta} \quad \text{similarly} \quad \csc^2 \theta = \frac{1}{\sin^2 \theta}$$

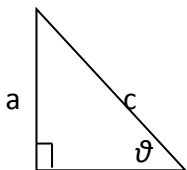
$$\text{So } 1 + \cot^2 \theta = \csc^2 \theta$$

**EXERCISE 44:**

- In the right triangle given below, find the value of  $\cot \theta$ .



- In the right triangle given below, work out the value of  $\sec \theta$ .



- Using  $\sin^2 \theta + \cos^2 \theta = 1$ , prove that  $\tan^2 x + 1 = \sec^2 x$

- Find the exact value of  $9 \sin^2 \theta + 9 \cos^2 \theta - 5$ .

- Simplify  $\frac{\sin^2 \theta + \cos^2 \theta - 5}{2}$ .

6. Find the value of  $\sin^2 \theta$  in terms of  $t$  if  $1 - \cos^2 \theta = t$ .
7. Prove that  $\sin x \cdot \sec x \cdot \cot x = 1$ .
8. Show that  $3\cos^2 \theta - 2 = 1 - 3\sin^2 \theta$ .
9. Show that  $\tan \theta \cos \theta = \sin \theta$ .
10. Solve the trig equation:  $\sin^2 \theta + \cos^2 30^\circ = 1$ , for  $0^\circ < \theta < 50^\circ$

### GRAPHS OF BASIC TRIGONOMETRIC EQUATIONS

| Trigonometric                                   | Shape | Period                           | Range<br>[y values] |
|-------------------------------------------------|-------|----------------------------------|---------------------|
| $y = \sin x$<br>$0^\circ \leq x \leq 360^\circ$ |       | $360^\circ$ or<br>$2\pi$ radians | $-1 \leq y \leq 1$  |
| $y = \cos x$<br>$0^\circ \leq x \leq 360^\circ$ |       | $360^\circ$ or<br>$2\pi$ radians | $-1 \leq y \leq 1$  |
| $y = \tan x$<br>$0^\circ \leq x \leq 360^\circ$ |       | $180^\circ$ or<br>$\pi$ radians  | $y \in \mathbb{R}$  |

Note these graphs are functions since domains are not repeated.

## TRANSFORMATION OF TRIGONOMETRIC GRAPHS

**Note:** General form of Transforming Trigonometric graphs

$$y = A \sin(Bx \pm C) \quad \text{or} \quad y = A \cos(Bx \pm C)$$

$$y = A$$

Sin  
Cos

$$Bx \pm C$$

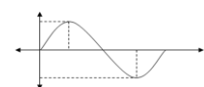
**Amplitude:**

The height / distance

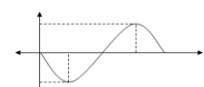
+ A means the graph is oriented as usual  
- A means that the graph is inverted

**Shape:**

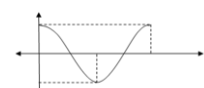
+ A Sin



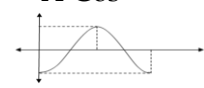
- A Sin



+ A Cos



- A Cos



**Period:**

B helps determine the *period* of the graph (the length of the interval needed for the graph of the function to start repeating itself).

$$\text{period} = \frac{360}{B} \text{ or } \frac{2\pi}{B}$$

C shifts the y-axis or the graph by  $\frac{C}{B}$  units.

$+\frac{C}{B}$  shifts the y-axis to the right or the graph moves by  $\frac{C}{B}$  units to the left.

$-\frac{C}{B}$  shifts the y-axis to the left or the graph moves by  $\frac{C}{B}$  units to the right

While sketching the graph, label clearly the y – intercept, period, amplitude and draw a complete smooth curve. You may use the table method but make sure the shape is complete.

**EXAMPLE 1:** A trigonometric function is defined by the equation  $y = 3 \sin 2\theta$ . Find the amplitude and the period of the function.

**General form**

$$y = A \sin(Bx \pm \theta)$$

Amplitude      period

Compare  $y = 3 \sin 2\theta$

Therefore Amplitude is 3 and the period is  $\frac{360}{2} = 180^\circ$  or  $\pi$  radians

**EXAMPLE 2:** Sketch the following graphs using the table method:

a)  $y = \sin x$

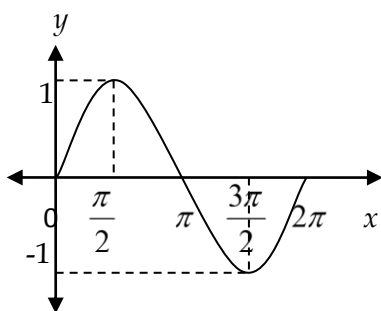
b)  $y = \sin\left(\frac{1}{2}x\right)$

c)  $y = \sin 2x$

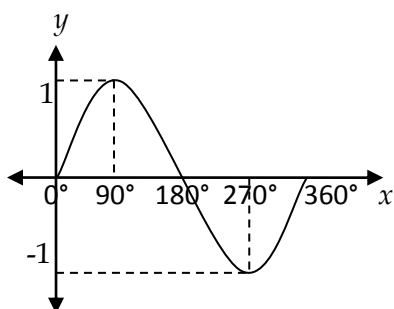
**Answer** [Either in degrees or in radians]

| $x$              | $y = \sin x$ | $(x, y)$               |
|------------------|--------------|------------------------|
| 0                | 0            | $(0, 0)$               |
| $\frac{\pi}{2}$  | 1            | $(\frac{\pi}{2}, 1)$   |
| $\pi$            | 0            | $(\pi, 0)$             |
| $\frac{3\pi}{2}$ | -1           | $(\frac{3\pi}{2}, -1)$ |
| $2\pi$           | 0            | $(2\pi, 0)$            |

The sine graph completes its shape from 0 to  $2\pi$ .

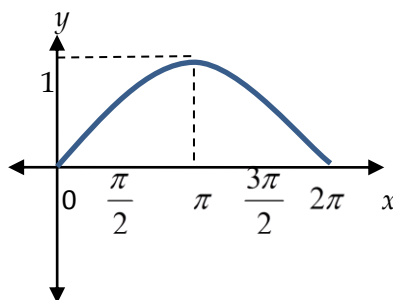


or

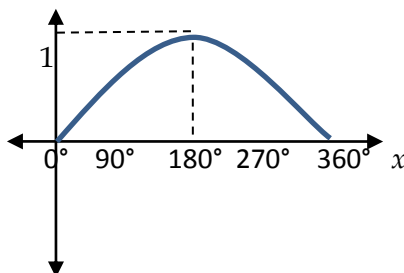


| $x$              | $y = \sin\left(\frac{1}{2}x\right)$ | $(x, y)$                |
|------------------|-------------------------------------|-------------------------|
| 0                | 0                                   | $(0, 0)$                |
| $\frac{\pi}{2}$  | 0.7                                 | $(\frac{\pi}{2}, 0.7)$  |
| $\pi$            | 1                                   | $(\pi, 1)$              |
| $\frac{3\pi}{2}$ | 0.7                                 | $(\frac{3\pi}{2}, 0.7)$ |
| $2\pi$           | 0                                   | $(2\pi, 0)$             |

Since the period is half, the half of sine graph is shown from 0 to  $2\pi$ .

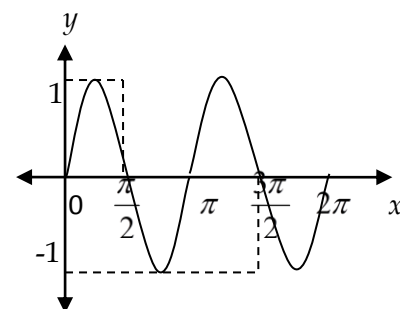


or

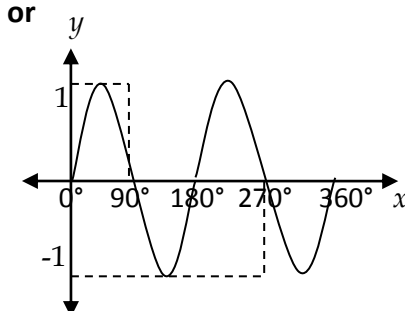


| $x$              | $y = \sin 2x$ | $(x, y)$               |
|------------------|---------------|------------------------|
| 0                | 0             | $(0, 0)$               |
| $\frac{\pi}{4}$  | 1             | $(\frac{\pi}{4}, 1)$   |
| $\frac{\pi}{2}$  | 0             | $(\frac{\pi}{2}, 0)$   |
| $\frac{3\pi}{4}$ | -1            | $(\frac{3\pi}{4}, -1)$ |
| $\pi$            | 0             | $(\pi, 0)$             |
| $\frac{5\pi}{4}$ | 1             | $(\frac{5\pi}{4}, 1)$  |
| $\frac{3\pi}{2}$ | 0             | $(\frac{3\pi}{2}, 0)$  |
| $\frac{7\pi}{4}$ | -1            | $(\frac{7\pi}{4}, -1)$ |
| $2\pi$           | 0             | $(2\pi, 0)$            |

Since the period is doubled, the two complete shape of sine graph is shown from 0 to  $2\pi$ .



or



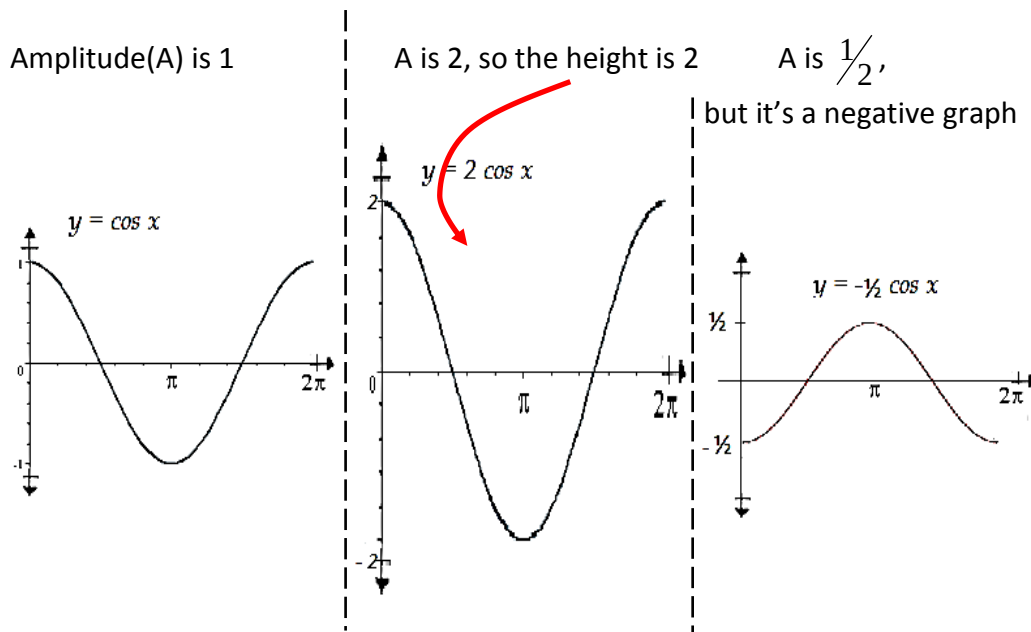
**EXAMPLE 3:** Sketch the following graphs:

a)  $y = \cos x$

b)  $y = 2 \cos x$

c)  $y = -\frac{1}{2} \cos x$

**Answer:** using the transformation method

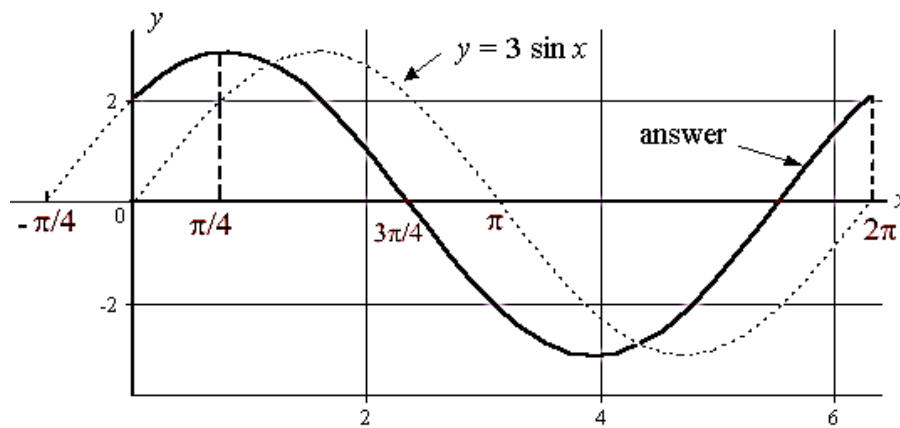


**EXAMPLE 4:** Sketch  $y = 3 \sin \left( x + \frac{\pi}{4} \right)$

A=3;

B: period=2 $\pi$ ;

C: Shift the graph  $\frac{\pi}{4}$  units to the left or Shift the y-axis by  $\frac{\pi}{4}$  units to the right



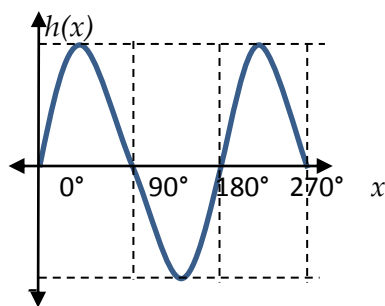
**EXERCISE 45:**

1. A trigonometric function is defined as  $f(x) = \sin \left( 2x + \frac{\pi}{4} \right)$  for  $0 \leq x \leq 2\pi$ .

(i) Write the period of the function  $f(x)$ .

(ii) Solve the equation  $\sin \left( 2x + \frac{\pi}{4} \right) = 0$

2. The graph of  $h(x)$  is shown below within the domain  $0^\circ \leq \theta \leq 270^\circ$ .  
Use it to answer the following questions.



- i) What is the period of the graph of  $h(x)$  shown above?
  - ii) What is the amplitude of  $h(x)$ ?
  - iii) Write down the equation of  $h(x)$ .
3. The two functions are given as  $y = \sqrt{3} \cos x$  and  $y = \sin x$ .
- i. Sketch the graph of  $y = \sin x$ .
  - ii. On the same pair of axes, sketch the graph of  $y = \sqrt{3} \cos x$ .
  - iii. Given that  $\frac{\sin \theta}{\cos \theta} = \tan \theta$ , find the  $x$  – coordinates of the points of intersection of the two graphs.
4. A trigonometric function is defined as  $f(x) = 3 \cos 2x$  where  $0 \leq x \leq 2\pi$ .
- (i) Write the period of the function  $f(x)$ .
  - (ii) Write down the minimum value of  $f(x)$ .
  - (iii) Sketch the graph of  $f(x)$ .
5. Sketch the following graphs using the domain as  $0 \leq \theta \leq 2\pi$ .
- |                                               |                                                |                                                |
|-----------------------------------------------|------------------------------------------------|------------------------------------------------|
| a) $y = 2 \cos\left(x + \frac{\pi}{2}\right)$ | b) $y = \sin\left(x - \frac{\pi}{2}\right)$    | c) $y = -\sqrt{2} \cos x$                      |
| c) $y = \cos(3x - \pi)$                       | e) $y = 3 \sin\left(2x + \frac{\pi}{2}\right)$ | f) $y = -3 \cos\left(x - \frac{\pi}{2}\right)$ |
6. Sketch the following graphs using the domain as  $0^\circ \leq x \leq 360^\circ$ .
- |                   |                               |                                           |
|-------------------|-------------------------------|-------------------------------------------|
| a) $y = 2 \cos x$ | b) $y = \cos(2x + 180^\circ)$ | c) $y = -3 \sin(2x + 90^\circ)$           |
| d) $y = -\sin 2x$ | e) $y = \sin 2x$              | f) $y = -\frac{1}{2} \cos(2x + 45^\circ)$ |

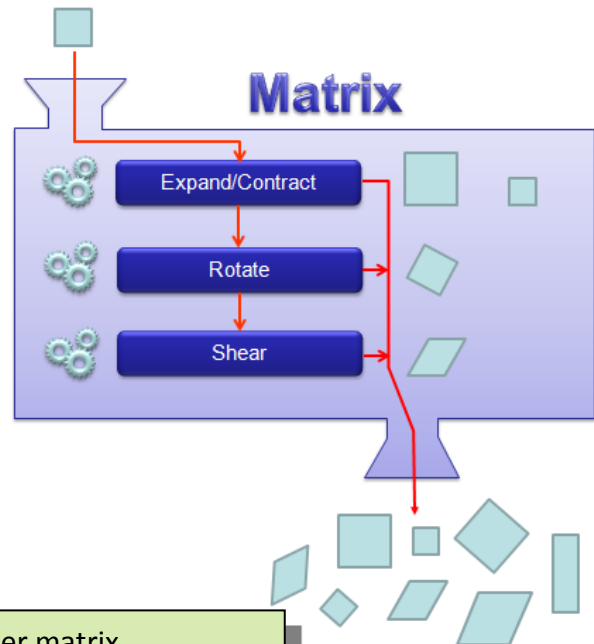
**Lesson of Life:** *The study of mathematics can satisfy a wide range of interests and abilities. It develops the imagination-* anonymous

# STRAND SIX

# MATRICES AND TRANSFORMATION GEOMETRY

SUB – STRAND 6.1

## Matrix Transformation



## CONTENT LEARNING OUTCOMES

- Image of a point after matrix transformations
- Finding transformations represented by matrices
- Invariant Properties
- Transformation matrix and its inverse
- The determinant as scale factor for area.
- Combination of transformations.

## INTRODUCTION TO MATRICES

**Note:** Recall that a matrix is nothing but a convenient way to organize data in rows and columns. Suppose you eat the following amount of fruits each day. The table below shows the number of fruits consumed for 3 days.

|        | Monday | Tuesday | Wednesday |
|--------|--------|---------|-----------|
| Apple  | 1      | 2       | 3         |
| Guava  | 2      | 1       | 1         |
| Banana | 1      | 1       | 2         |
| mango  | 3      | 3       | 1         |

This can be re-organized in a matrix below in four rows and 3 columns:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \\ 3 & 3 & 1 \end{pmatrix}$$

The matrix is a  $4 \times 3$  matrix and is read “4 by 3”. This is called the **order** of a matrix. It does not mean multiplication.

Another way to represent the information in a table is

|           | Apple | Guava | Banana | Mango |
|-----------|-------|-------|--------|-------|
| Monday    | 1     | 2     | 1      | 3     |
| Tuesday   | 2     | 1     | 1      | 3     |
| Wednesday | 3     | 1     | 2      | 1     |

As a matrix this can be written as:

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 1 \end{pmatrix}$$

It is a  $3 \times 4$  matrix.

Though it represents the same information, but these two matrices are not equal.

Two matrices are **equal** when they have the **same dimensions** and same **corresponding elements**.

## TRANSFORMING POINTS

**Note:** Transformation by Matrices is useful for modeling, simulations and computer graphics.

- Before actually looking at the transformation, some knowledge of **multiplication of  $2 \times 2$**  matrices is essential in order to transform points in a plane.

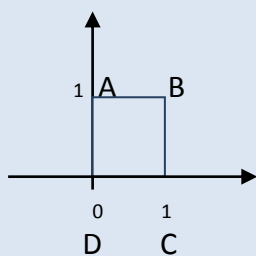
Two matrices may be multiplied if the number of columns of the first matrix is equal to the number of the rows of the second matrix. Moreover, matrices are multiplied row by column, i.e. you take the first row of the matrix A and multiply with the first column of the matrix B.

- The ordered pairs [coordinates,  $(x,y)$ ] can be represented by  $2 \times 1$  column matrix:

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

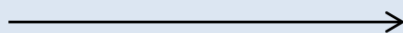
- Finally, to transform a point, i.e. from Object point (A) to an image point [symbol A']:
  - Write the transformation matrix first.
  - Change the coordinate points and write it as a column matrix on the right side.
  - Perform matrix multiplication.
  - (Transformation matrix)  $\times$  (point matrix) = image point
  - Write as coordinates  $(x,y)$ .

- **A unit square** has side of length 1 unit. Consider the diagram given below: Let ABCD be the points of the corners.



The coordinates of the corners of a unit square are  
(0, 0) (0, 1) (1, 1) (1, 0)

- Transformation



**EXAMPLE 1:** Find the image points of the vertices of a unit square under the transformation matrix  $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ .

**Steps:**

➡ Write the transformation matrix first:  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

➡ Change the coordinate points and write as column matrix:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

➡ Perform matrix multiplication.

Note: You may do it separately or take all points at once.

**Let's find image point of (1,1):** Identify rows of first matrix columns of second matrix,

$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \end{array} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{l} \text{Column 1} \\ \text{Column 2} \end{array} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Matrices are multiplied row by column

|                               |
|-------------------------------|
| $2 \times 1 + 0 \times 1 = 2$ |
| $0 \times 1 + 2 \times 1 = 2$ |

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

➡ Hence, the image point of (1,1) as coordinates is **(2,2)**.

**Now, let's find the image points of the other three points all at once:**

$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \end{array} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{l} \text{C1} \\ \text{C2} \\ \text{C3} \end{array} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

| Column 1                           | Column 2                           | Column 3                           |
|------------------------------------|------------------------------------|------------------------------------|
| $2 \times 0$<br>$+ 0 \times 0 = 0$ | $2 \times 0$<br>$+ 0 \times 1 = 0$ | $2 \times 1$<br>$+ 0 \times 0 = 2$ |
| $0 \times 0$<br>$+ 2 \times 0 = 0$ | $0 \times 0$<br>$+ 2 \times 1 = 2$ | $0 \times 1$<br>$+ 2 \times 0 = 0$ |

$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \end{array} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

Hence, the image points are **(2,2), (0,0), (0,2), (2,0)**.

**EXAMPLE 2:** The image of point P (m,-3) under transformation by matrix

$\begin{pmatrix} 1 & 2 \\ 3 & n \end{pmatrix}$  is Q (-2,3). Find the values of m and n.

**To find transformation image point:**

**Steps:**

➡ Write the  $2 \times 2$  matrix first:

$$\begin{pmatrix} 1 & 2 \\ 3 & n \end{pmatrix}$$

➡ Change the coordinate points and write as column matrix:

$$\begin{pmatrix} 1 & 2 \\ 3 & n \end{pmatrix} \begin{pmatrix} m \\ -3 \end{pmatrix}$$

➡ Perform matrix multiplication:

$$\begin{array}{l} \text{Row} \\ \text{Row} \end{array} \begin{array}{l} 1 \\ 2 \end{array} \begin{array}{l} \begin{pmatrix} 1 & 2 \\ 3 & n \end{pmatrix} \begin{pmatrix} m \\ -3 \end{pmatrix} \end{array} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

|                                      |
|--------------------------------------|
| $1 \times m + 2 \times -3 = m - 6$   |
| $3 \times m + n \times -3 = 3m - 3n$ |

$$= \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Equate the Row 1 equations Row 2 equations separately. Solve.

$$m - 6 = -2$$

$$m - 6 + 6 = -2 + 6$$

$$\therefore m = 4$$

Substitute in place of m:

$$3(4) - 12 - 3n = 3$$

$$\Rightarrow 12 - 3n = 3$$

$$\frac{-3n}{-3} = \frac{-9}{-3}$$

$$n = 3$$

Therefore  $m = 4$   $n = 3$ .

**Attempt the following questions to check whether you have understood.**

The coordinate points are given as **P (1,1), Q (3,2), R (3,1)**.

Find the coordinates of P' Q' and R', under the transformation by the following matrices:

a)  $M = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

b)  $N = \begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}$

c)  $P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

d)  $Q = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

## FINDING THE TRANSFORMATION WHICH A 2 X 2 MATRIX REPRESENTS

**Note:** To describe a transformation:

- Draw the **object** by plotting the coordinates given and join to form a shape.
- Multiplying with the matrix to find the image points denoted by A'.
- Plot the image points and join to form the shape of the **image**
- Describe the transformation.

**Types of transformation:**

1. Rotation
  2. Enlargement
  3. Shear
  4. Reflection
- If no points are given, in order to describe a transformation, a unit square can be considered and the image of each vertex of the square is calculated by matrix multiplication.
  - A point (or set of points) which remains unchanged after a transformation is said to be **invariant**.

Notice that the origin (0, 0) always remains invariant (unchanged) under a  $2 \times 2$  transformation matrix.

- The **determinant** of a transformation matrix gives the **area scale factor**.  
i.e. If A is a transformation matrix,  $|A|$  is the scale factor for area.  
Area scale factor = determinant  
So area of image = determinant  $\times$  Area of object
- **Inverse Matrix:** If a transformation represented by matrix M maps a point P to its image point, P', then P' will map back to its original position, P under the transformation by  $M^{-1}$ . The general formula for inverse of a matrix is

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M^{-1} = \frac{1}{|M|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

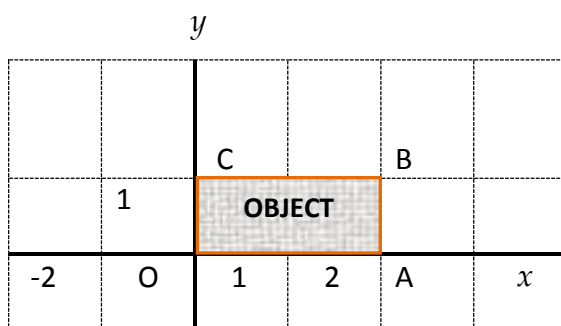
## 1. Rotation of $\pm \theta^\circ$ (angle in degrees)

### Note:

- In mathematics context, Rotation is a transformation in which the objects are rotated by a fixed angle,  $\theta$ , about the centre. It can rotate either clockwise ( $-\theta$ ) or anti-clockwise ( $\theta$ ).
- In rotation, the centre is the only invariant point.
- Invariant features of rotation are length, angle size and area.

### EXAMPLE 3:

The diagram below shows rectangle OABC



The rectangle is transformed by matrix  $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  to give the image  $O' A' B' C'$ .

The coordinates of  $O'$  is (0,0).

- Find the coordinates of  $A'$ ,  $B'$ ,  $C'$ , the images of A,B C respectively under transformation by matrix P.
- On a pair of axes, draw accurately the object and its image.
- Describe fully the transformation given by matrix P

### Answers:

**C1 C2 C3**

i.

**A B C**

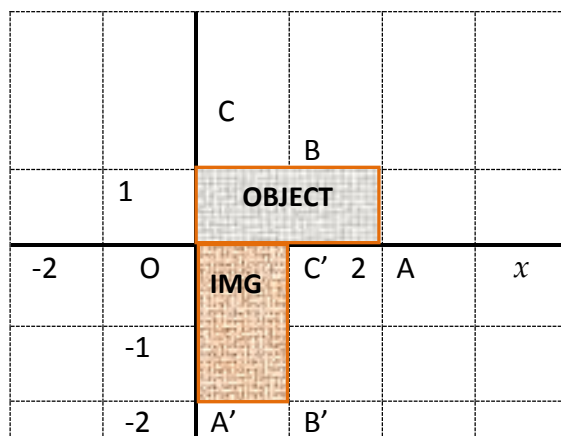
$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \end{array} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

|                                      |                                      |                                     |
|--------------------------------------|--------------------------------------|-------------------------------------|
| $0 \times 2$<br>$+ 1 \times 0 = 0$   | $0 \times 2$<br>$+ 1 \times 1 = 1$   | $0 \times 0$<br>$+ 1 \times 1 = 1$  |
| $-1 \times 2$<br>$+ 0 \times 0 = -2$ | $-1 \times 2$<br>$+ 0 \times 1 = -2$ | $-1 \times 0$<br>$+ 0 \times 1 = 0$ |

$$= \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus,  $A' (0, -2)$ ,  $B' (1, -2)$   $C' (1,0)$

ii.



iii. Rotation of  $-90^\circ$  or  $270^\circ$  about the origin.

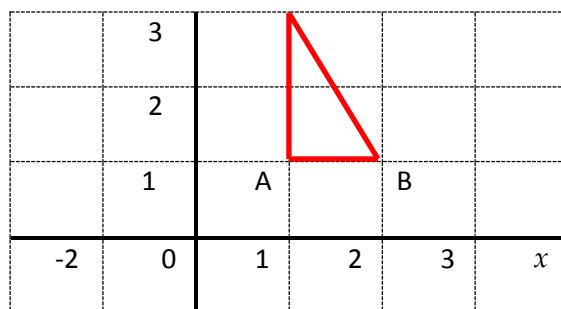
## 2. Enlargement by a Scale Factor, $k$

**Note:** The general matrix for an enlargement is  $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ , where  $k$  is the length scale factor and  $k^2$  (*determinant*) is the area scale factor.

- In **enlargement**, the centre is the only invariant point.
- Angle size is invariant under enlargement.

e.g.  $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$  represents an enlargement with scale factor 4.

**EXAMPLE 4:** The diagram below shows triangle ABC



The triangle is transformed by matrix  $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  to give triangle  $A'B'C'$ .

- Find the coordinates of  $A'$ ,  $B'$ ,  $C'$ , the images of A, B, C respectively under transformation by matrix M.
- Describe fully the transformation given by matrix M.

**Answers:**

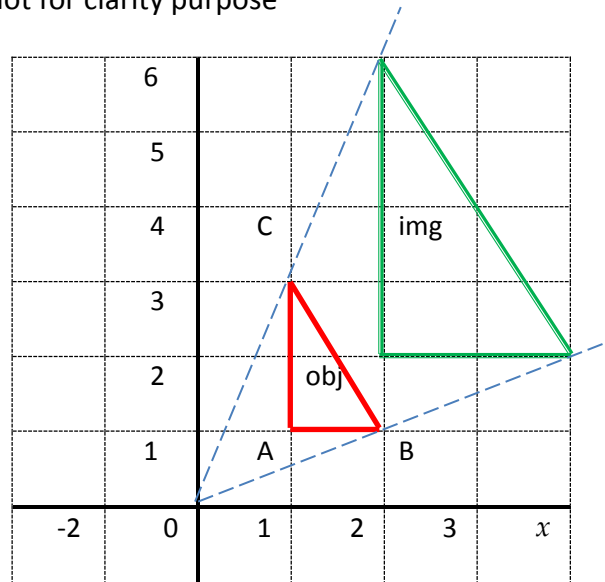
i. 
$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} =$$

|                                    |                                    |                                    |
|------------------------------------|------------------------------------|------------------------------------|
| $2 \times 1$<br>$+ 0 \times 1 = 2$ | $2 \times 2$<br>$+ 0 \times 1 = 4$ | $2 \times 1$<br>$+ 0 \times 3 = 2$ |
| $0 \times 1$<br>$+ 2 \times 1 = 2$ | $0 \times 2$<br>$+ 2 \times 1 = 2$ | $0 \times 1$<br>$+ 2 \times 3 = 6$ |

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & 6 \end{pmatrix}$$

Thus  $A' (2,2)$ ,  $B' (4,2)$   $C' (2,6)$ .

ii. First plot for clarity purpose



Enlargement by the scale factor 2 with the origin as the centre.

### 3. Shear

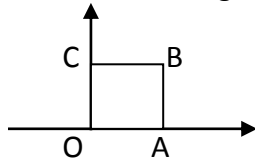
**Note:** A shear is a transformation which changes the shape of an object but not its area.

- **Invariant line** is the fixed axis in a **shear**. Thus, Shear is a transformation which moves points parallel to a given line. The **fixed line** is called the **axis of shear**. **Shear factor** is the **distance moved** by a point that is one unit from the fixed axis.
- The general matrix for a shear parallel to:
  - the x-axis is:  $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ ,
  - the y-axis is:  $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$  where  $k$  is the shear factor.

The Properties of shear

1. The points on the fixed axis are invariant
2. Area is invariant under a shear

**EXAMPLE 5:** The diagram given below shows the unit square OABC.



The unit square OABC is transformed by matrix  $M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

- Find the coordinates of  $O'A'B'C'$ , the images of O, A, B, C under the transformation by matrix M.
- Which point(s) is/are invariant?
- Describe fully the transformation given by matrix M.
- Find the single matrix that will map  $O'A'B'C'$  back into OABC.

**Answers:**

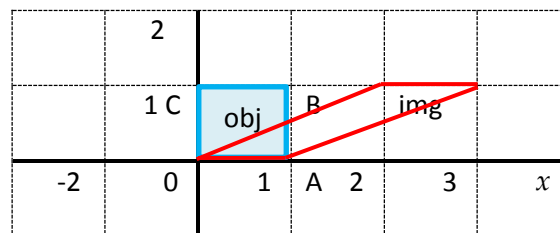
i.  $\begin{matrix} O & A & B & C \\ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{matrix} =$

|                                                                                |                                                                                |                                                                                |                                                                                |
|--------------------------------------------------------------------------------|--------------------------------------------------------------------------------|--------------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| <del><math>1 \times 0</math></del><br><del><math>+ 2 \times 0 = 0</math></del> | <del><math>1 \times 1</math></del><br><del><math>+ 2 \times 0 = 1</math></del> | <del><math>1 \times 1</math></del><br><del><math>+ 2 \times 1 = 3</math></del> | <del><math>1 \times 0</math></del><br><del><math>+ 2 \times 1 = 2</math></del> |
| <del><math>0 \times 0</math></del><br><del><math>+ 1 \times 0 = 0</math></del> | <del><math>0 \times 1</math></del><br><del><math>+ 1 \times 0 = 0</math></del> | <del><math>0 \times 1</math></del><br><del><math>+ 1 \times 1 = 1</math></del> | <del><math>0 \times 0</math></del><br><del><math>+ 1 \times 1 = 1</math></del> |

$\begin{matrix} O' & A' & B' & C' \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{matrix}$

Thus  $O' (0,0)$ ,  $A' (1,0)$ ,  $B' (3,1)$ ,  $C' (2,1)$ .

- Points that do not change are O (0, 0) A (1, 0).
- For clarity purpose, show on the axes:



Thus, shear parallel to x – axis with a shear factor of 2 units.

iv.

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

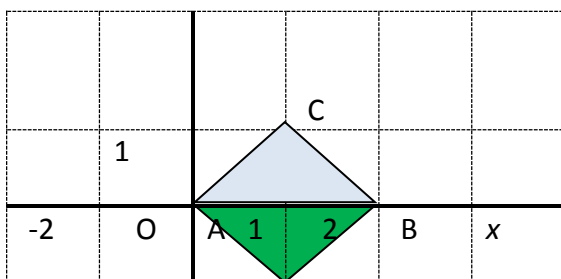
$$M^{-1} = \frac{1}{|M|} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \frac{1}{1 \times 1 - 0 \times 2} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

#### 4. Reflection

##### Note:

- Under **Reflection**, all points on the **mirror line** are invariant. Other invariant features are length, angle, size and area.
- The general matrix for Reflection in:
  - the x-axis is:  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,
  - the y-axis is:  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

**EXAMPLE 6:** The object has points A(0,0), B(2,0) C (1,1) while the image points A'(0,0), B'(2,0) C' (1,-1). What transformation does this represent?



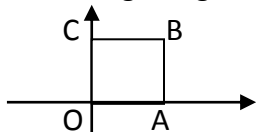
##### Reflection in x – axis

When multiplying by this matrix, the  $x$  coordinate remains unchanged, but the  $y$  coordinates changes sign.

##### EXERCISE 46:

- A triangle has vertices at A(1,0), B(1,3) C (2,0). On a pair of axes, draw  $\triangle ABC$ . Transform  $\triangle ABC$  using the matrix  $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Label the images  $A'B'C'$  describe the transformation.

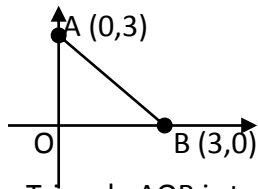
- The diagram given below shows the unit square OABC.



The unit square OABC is transformed by matrix  $N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

- Find the coordinates of O'A'B'C', the images of O, A, B and C under the transformation by matrix N.
- Describe fully the transformation given by matrix N.

3. The diagram given below shows triangle AOB.



Triangle AOB is transformed by matrix  $M = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$  to give Triangle A'O'B'.

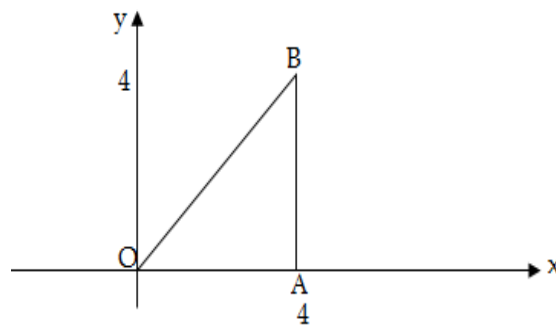
- Find the coordinates of A' and B', the images of A and B under the transformation by matrix M.
- Describe fully the transformation given by matrix M.

4. A triangle has vertices at A(1,0), B(1,3) C (2,0). On a pair of axes, draw  $\triangle ABC$ . Transform  $\triangle ABC$  using the matrix  $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ . Label the images A'B'C' describe the transformation.

5. A triangle has vertices at A(1,0), B(1,3) C (2,0). On a pair of axes, draw  $\triangle ABC$ . Transform  $\triangle ABC$  using the matrix  $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Label the images A'B'C' describe the transformation.

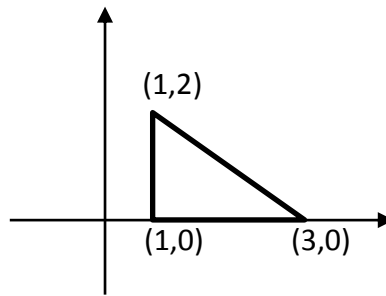
6. Figure OAB in the diagram given below is an isosceles triangle. It is transformed by the matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



- Find the coordinates of O' A' B'.
- Sketch the points O' A' B'.
- Find the single matrix that will map O'A'B'C' back into OABC.

7. The triangle OAB is to be transformed by Matrix M where  $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Copy the diagram given and on the same axes mark points  $O'$ ,  $A'$ ,  $B'$  after the transformation.

- Describe the transformation represented by M?
- Another transformation of the plane is given by  $N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$   
Find and draw the image of  $O'$ ,  $A'$ ,  $B'$  as  $O''$ ,  $A''$ ,  $B''$  respectively.
- What single transformation would map OAB to  $O''$ ,  $A''$ ,  $B''$ ?

8. A unit square is transformed by matrix  $N = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

- Find the coordinates of the images of the vertices under the transformation by matrix N.
- Describe fully the transformation given by matrix N.

## **COMBINED TRANSFORMATIONS**

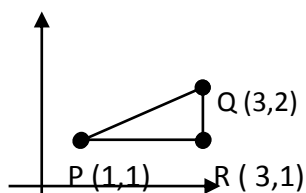
**Note:**

**Combined transformation:** Sometimes you need to transform the first image point to get the second image point with another matrix. These are known as **double image points**,  $A''$ .

**Finding a single matrix for combined transformation:** If point P is transformed by matrix A to get  $P'$  and further transformed by matrix B to get  $P''$  then

**a single matrix that will map  $P'$  directly to  $P''$  is the matrix  $B.A$**

**EXAMPLE 5:** The diagram given below shows triangle PQR.



Triangle is transformed by matrix  $M = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ .

- Find the coordinates of  $P'$ ,  $Q'$  and  $R'$ , under the transformation by matrix  $M$ .
- Describe fully the transformation given by matrix  $M$ .
- Find the area of triangle  $P'Q'R'$ .
- Triangle  $P'Q'R'$  is further transformed to triangle  $P''Q''R''$  by another matrix  $N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Find the single matrix that will map triangle  $ABC$  onto  $A''B''C''$ .

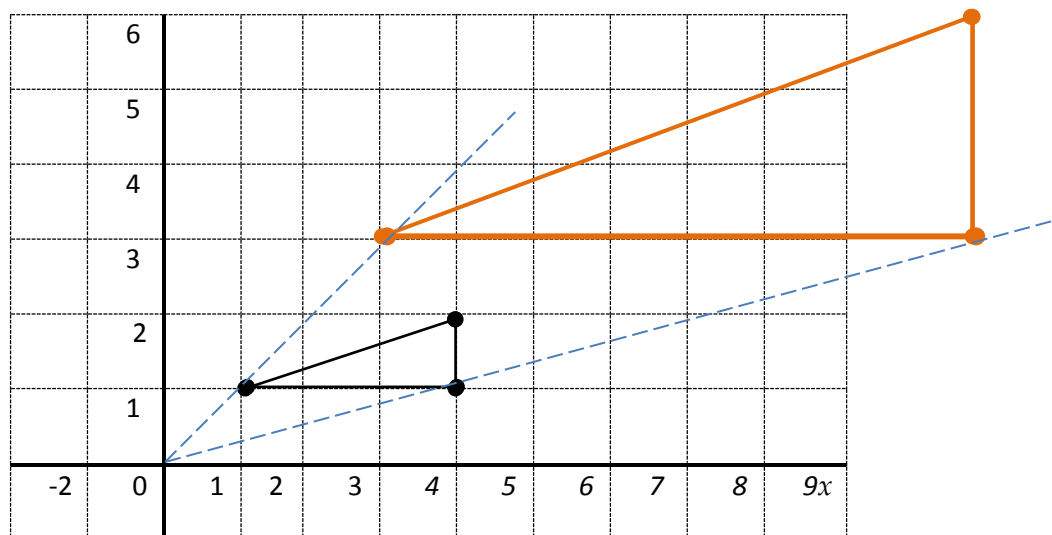
**Answers:**

i.  $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 \\ 1 & 2 & 1 \end{pmatrix} =$

|                                                            |                                                            |                                                            |
|------------------------------------------------------------|------------------------------------------------------------|------------------------------------------------------------|
| $3 \times 1$<br><del><math>+ 0 \times 1 = 3</math></del>   | $3 \times 3$<br><del><math>+ 0 \times 2 = 9</math></del>   | $3 \times 3$<br><del><math>+ 0 \times 1 = 9</math></del>   |
| <del><math>- 0 \times 1</math></del><br>$+ 3 \times 1 = 3$ | <del><math>- 0 \times 3</math></del><br>$+ 3 \times 2 = 6$ | <del><math>- 0 \times 3</math></del><br>$+ 3 \times 1 = 3$ |

$$= \begin{pmatrix} 3 & 9 & 9 \\ 3 & 6 & 3 \end{pmatrix}$$

Thus  $P' (3,3)$ ,  $Q' (9,6)$   $R' (9,3)$ .



- Enlargement by the scale factor 3 with centre  $(0,0)$ .
- Area scale factor = determinant =  $3 \times 3 - 0 \times 0 = 9$

$$\begin{aligned}\text{Thus Area of image} &= \text{Area scale factor} \times \text{Area of an object} \\ &= 9 \times \frac{1}{2} \text{ base} \times \text{height} = 9 \times \frac{1}{2} \times 2 \times 1 = 9 \text{ unit}^2\end{aligned}$$

iv.

$$N \times M = \begin{matrix} & \begin{matrix} \text{C1} & \text{C2} \end{matrix} \\ \begin{matrix} \text{Row 1} \\ \text{Row 2} \end{matrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{matrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

|                        |                         |
|------------------------|-------------------------|
| 0 × 3<br>+ − 1 × 0 = 0 | 0 × 0<br>+ − 1 × 3 = −3 |
| 1 × 3<br>+ 0 × 0 = 3   | 1 × 0<br>+ 0 × 3 = 0    |

$$= \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$$

#### EXERCISE 47:

1. Given  $S = \begin{pmatrix} -4 & 1 \\ 7 & -2 \end{pmatrix}$   $G = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ .

Find:

- $G \times S$
- $G^{-1}$

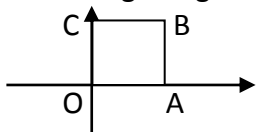
2. A square has vertices at A (0, 0), B(0, 2), C(2, 2) and D(2, 0).

- On a pair of axes, draw the square.
- Transform the square using the matrix  $M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ . Label the images as  $A', B', C', D'$ . Describe fully the transformation given by matrix  $M$ .
- Transform square  $A'B'C'D'$  with matrix  $P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  label the image  $A''B''C''D''$ . Describe fully the transformation given by matrix  $P$ .
- Find the matrix which will transform square ABCD directly onto  $A''B''C''D''$ .

3. Object P having an area of 'A' square unit is transformed by the matrix  $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ .

Find the area of the image  $P'$  give answer in terms of ' $A$ '.

4. The diagram given below shows the unit square OABC.



The unit square OABC is transformed by matrix  $M = \begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix}$ . Work out the area of the image.

5. A square, having an area of  $K$  units is transformed by matrix  $Q = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$ .

What is the area of the transformed square? [Write in terms of  $K$ .]

6. A triangle has vertices  $A(1, 0)$ ,  $B(1, 3)$  and  $C(2, 0)$ .

i. On a pair of axes, draw  $\triangle ABC$ .

ii. Transform  $\triangle ABC$  using the matrix  $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Label the images as  $A', B', C'$ . Describe fully the transformation given by matrix  $M$ .

iii. Transform  $\triangle A'B'C'$  with matrix  $P = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$  label the image  $A''B''C''$ .

Describe fully the transformation given by matrix  $P$ .

iv. Find the matrix which will transform  $\triangle ABC$  directly onto  $A''B''C''$ .

7. A triangle has vertices  $O(0,0)$ ,  $P(1,0)$  and  $Q(2,3)$ . It has to be transformed by the matrix  $M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

- Find the coordinates for  $O'$ ,  $P'$ ,  $Q'$  which are the images of  $O$ ,  $P$  and  $Q$  respectively under matrix  $M$  plot on the same pair of axes in the answer book.
- Describe the transformation represented by matrix  $M$ .
- What is the area of the transformed triangle?

**Lesson of Life:** *Mathematics has beauty romance. It's not a boring place to be, the mathematical world. It's an extraordinary place; it's worth spending time there ~ Marcus du Sautoy*

# STRAND SEVEN

# STATISTICS

## SUB – STRAND 7.1

## *Statistical Analysis*

### CONTENT LEARNING OUTCOMES

- Introduction
- Measures of centre
- Measures of spread including standard deviation of both grouped and ungrouped data.
- Adding or multiplying a constant to every value
- Interpret information from different types of graph



## INTRODUCTION

**Note:** **Statistics** is the collection, display and analysis of data. Some important terms that can be useful are:

- A **population** considers **every** member of the group for instance "all students in a school". Sometimes it becomes very difficult to involve the whole population because the
  1. whole population may not be accessible.
  2. study may not be economical.
  3. study may take a lot of time etc.

The problem is solved by picking a smaller group from the population for the survey.

This group is called a sample.

- A **Sample** is when we collect data just for **selected members** of the group. It is a subset of the population. This depends **on what actually you want to** find out.

For instance, if a school has Years 9 to 13 and a survey has to be conducted to decide on some important school matter, then taking students randomly from all the forms is a good sample.

- **Random sample:** a sample selected where each element has an equal chance of selection.

For example, if 5 students are picked from your form to represent the form in a meeting then 5 names picked from a hat/glass is a fair sample.

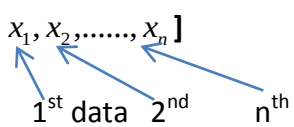
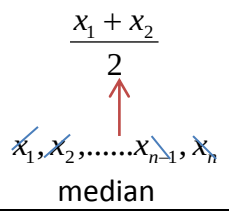
**Data** is a collection of facts, such as numbers, words, measurements, observations or even just descriptions of things.

## MEASURES OF CENTRAL TENDENCY

**Note:** A measure of **central tendency** (also referred as measures of centre or central location) attempts to describe a whole set of data with a **single value** that represents the **middle or centre** of its distribution.

There are three main measures of central tendency: the **mode**, the **median** and the **mean**. Each of these measures describes a different indication of the typical or central value in the distribution.

Related formulae:

| Measures of central tendency                                                                                                                                                                                                   | UNGROUPED DATA – individual scores [ $x_1, x_2, \dots, x_n$ ]<br>                                                                                                                                                                                       | GROUPED DATA – frequency tables<br>$f$ – frequency<br>$x$ – data                                                  |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------|
| <b>Mean / Average</b><br>Note <b>Sigma</b> sign $\Sigma$ indicates the sum.<br>Mean is the most common of the three.<br>The symbol " $\mu$ " is used for population mean.<br>The symbol " $\bar{x}$ " is used for sample mean. | $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} \text{ or } \frac{\text{Total}}{\text{number}} \text{ or } \bar{x} = \frac{\sum x}{n}$                                                                                                                                                                                                      | $\bar{x} = \frac{\sum fx}{\sum f} \text{ or } \frac{\sum fx_m}{\sum f}$<br>$x_m$ – midpoint of the class interval |
| <b>Median – middle score</b><br>The median is the midpoint of a distribution when all the scores are placed in order                                                                                                           | Arrange score in an ascending order. Locate the middle score by cancelling the data from both left/right till you reach to the number in the middle.<br>For even number of scores, it is the middle of two numbers, i.e.<br>$\frac{x_1 + x_2}{2}$<br> | $\text{median score corresponding to the cumulative frequency of } \frac{\sum f}{2}$                              |
| <b>Mode – most common number</b>                                                                                                                                                                                               | Identify the score that is occurring most frequently                                                                                                                                                                                                                                                                                     | Score that has the highest frequency                                                                              |

## FINDING SPREAD

**Note:** Measures of spread describe how similar or varied the set of observed values are for a particular variable (data item). Measures of spread include the **range**, **quartiles** [spread of a data set by breaking the data set into quarters], **interquartile range** and **standard deviation**.

Related formulae:

| Measures of Spread                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | UNGROUPED DATA – individual scores                                       | GROUPED DATA – frequency tables                                                                                                                                                                                                                                                                            |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Range (R)</b> highest score minus the lowest score                                                                                                                                                                                                                                                                                                                                                                                                                              | $R = H_s - L_s$<br>the difference between the highest and lowest scores. | $R = H_s \text{ of } x - L_s \text{ of } x$                                                                                                                                                                                                                                                                |
| <p>The median divides the data into two equal halves. The median of these two halves are called upper and lower quartiles</p> <ul style="list-style-type: none"> <li>To find quartiles, first find the median. Put this value aside.</li> <li>This then gives two sets of values of the same size- a low group and a high group.</li> <li>The <b>lower quartile</b> is the median of the low group.</li> <li>The <b>upper quartile</b> is the median of the high group.</li> </ul> |                                                                          | <p><b>Quartiles from Cumulative frequency Graphs:</b></p> <p>To find the UQ, come across horizontally from 75% of the total frequency.<br/>For the lower quartile, use 25%.</p> $\frac{1}{4} \text{ of } 32 = 8$ $\therefore \text{LQ} = 28$ $\frac{3}{4} \text{ of } 32 = 24$ $\therefore \text{UQ} = 33$ |
| <b>interquartile range (IQ)</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                    | $\text{IQ} = \text{UQ} - \text{LQ}$                                      |                                                                                                                                                                                                                                                                                                            |
| <b>Semi-interquartile range</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                    | $\text{Semi IQ} = \frac{1}{2} \text{ of IQ or } \frac{\text{IQ}}{2}$     |                                                                                                                                                                                                                                                                                                            |

## STANDARD DEVIATION OF UNGROUPED DATA

The **standard deviation** (symbol  $\sigma$ ) of a set of numbers is a measure of their **spread** about the mean. The square of the standard deviation ( $\sigma^2$ ) is called the **variance**.

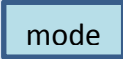
- Variance :  $\sigma^2 = \frac{\sum(x-\bar{x})^2}{n}$ , so  $\sigma$  (standard deviation) =  $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$

where  $\bar{x}$  is mean ,  $n$  is number of data values,  $x$  is each data value.  
 $\sum(x - \bar{x})^2$  means subtract the mean from each data value, square and finally add up the resulting values.

- Using calculator: If you are using Casio fx-82 MS calculator, use the procedure for data

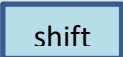
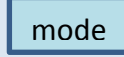
$$x_1, x_2, x_3, x_4, x_5 \dots x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}, x_n$$

**To write in SD mode**

- Press  

On the screen you should see SD.

**To clear memory( You must always clear the memory for each new set of data.)**

- Press   

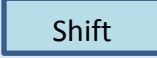
On the screen you should **Stat clear**. Press the = button.

- Press        

(press each number together with M+ button till the last score)

On the screen you should see 'n = last number' . This means that altogether that many data entry has been entered in your calculator

- Press

- For standard deviation, Press    
- For mean, Press    

Note: the other calculators may have slightly different features.

**EXAMPLE 1:** The mean weight of a group of 6 students is 62kg. Another student whose weight is 52kg is to be included in the group. Find the mean weight of 7 students.

Mean is 62kg  
n is 6  
Total?

$$\begin{aligned} \text{Mean} &= \frac{\text{Total}}{n} \\ 62 &= \frac{\text{Total}}{6} \\ \therefore \text{Total} &= 62 \times 6 = 372 \end{aligned}$$

If another student's weight of 52kg is added, then Total will increase by 52 as well as the number of students will now be 7, i.e. n is 7.

$$\begin{aligned} \text{Mean} &= \frac{\text{Total}}{n} \\ \bar{x} &= \frac{\text{Total} + 52}{7} = \frac{372 + 52}{7} \\ \therefore \bar{x} &= 60.571 \approx 60.6 \text{kg} \end{aligned}$$

Thus, the mean weight of 7 students is 60.6kg.

**EXAMPLE 2:** Use the first formula to find the standard deviation of **5,6,10,4,7,2,8,8,5**

$$\begin{aligned} \bar{x} &= \frac{x_1 + x_2 + \dots + x_n}{n} \\ &= \frac{5 + 6 + 10 + 4 + 7 + 2 + 8 + 8 + 5}{9} \\ &= \frac{55}{9} \\ &= 6.11 \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\ &= \sqrt{\frac{(5 - 6.11)^2 + (6 - 6.11)^2 + (10 - 6.11)^2 + \dots + (5 - 6.11)^2}{9}} \\ &= \sqrt{\frac{46.889}{9}} \\ &= \sqrt{5.2098} = 2.28 \end{aligned}$$

**EXAMPLE 3:** Use a calculator to find the mean and standard deviation.

- a) 5,6,10,4,7,2,8,8,5  
b) 102,324,627,231

- Put calculator mode on SD and clear the memory.

Press each number with M+ sign.

a) Press

|   |    |   |    |    |    |   |    |
|---|----|---|----|----|----|---|----|
| 5 | M+ | 6 | M+ | 10 | M+ | 4 | M+ |
| 7 | M+ | 2 | M+ | 8  | M+ | 8 | M+ |
| 5 | M+ |   |    |    |    |   |    |

|       |   |   |   |                  |
|-------|---|---|---|------------------|
| Shift | 2 | 1 | = | $\bar{x} = 6.11$ |
| Shift | 2 | 2 | = | $\sigma = 2.28$  |

- Once again clear the memory

Press

|     |    |     |    |     |    |     |    |
|-----|----|-----|----|-----|----|-----|----|
| 102 | M+ | 324 | M+ | 627 | M+ | 231 | M+ |
|-----|----|-----|----|-----|----|-----|----|

|       |   |   |   |                   |
|-------|---|---|---|-------------------|
| Shift | 2 | 1 | = | $\bar{x} = 321$   |
| Shift | 2 | 2 | = | $\sigma = 193.46$ |

**EXAMPLE 4:** The following are the amount of weekly salaries received by the 10 employees of a small company.

**\$120, \$132, \$144, \$180, \$180, \$204, \$204, \$204, \$228, \$192**

Calculate the

- |                       |                   |                   |
|-----------------------|-------------------|-------------------|
| a) Mean               | b) Mode           | c) Range          |
| d) Median             | e) lower quartile | f) Inter-quartile |
| g) Standard deviation |                   |                   |

**Answers: It is an ungrouped data**

a) Mean

$$\begin{aligned}\bar{x} &= \frac{x_1 + x_2 + \dots + x_n}{n} \\ &= \frac{\$120 + \$132 + \dots + \$192}{10} \\ &= \frac{\$1788}{10} \\ &= \$178.80\end{aligned}$$

b) Mode

\$204

c) Range

$$228 - 120 = \$108$$

Arrange the data in ascending order to find median and the quartiles

\$120, \$132, \$144, \$180, \$180, \$192, \$204, \$204, \$204, \$228

↑  
LQ
↑  
Median
↑  
UQ

d) Median

e). LQ = \$144

f). UQ = \$204

$$\begin{aligned}&= \frac{\$180 + \$192}{2} \\ &= \frac{\$372}{2} = \$186\end{aligned}$$

g). s.d = 33.71 (using the calculator)

#### Exercise 48:

a) A study is conducted on how the weight of a person affects the health. These are the weights of 20 girls of height [150 – 155cm]

42.5   45.3   45.7   51.0   55.1   55.7   55.7   59.1   59.5   59.7  
61.0   62.5   65.2   67.3   69.8   71.0   74.6   75.2   79.3   82.0

1. Calculate the

b) Mean

b) Mode

c) Range

e) Median

e) lower quartile

f) Inter-quartile

g) Standard deviation

2. Do a research and find out the desired weight and BMI of the required heights.

3. What are the health issues if you are under weight or over weight?

b) For each of the following data given below, use calculator to find the mean and Standard deviation.

1. {1,3,3,5}

2. {7,7,7,7}

3. {30,30,30,30}

4. {2,6,8,12}

5. { 7,6,8,7,5,9,8,6,7,10}

6. {2,3,0,3,2,1,4,3}

7. {2, 2, 4, 6, 6, 8, 10, 10}

8. {2, 2, 4, 4, 8, 8, 10, 10}

## STANDARD DEVIATION FROM FREQUENCY TABLES

**Note:** Recall that Standard deviation is the square root of the variance.

Thus the formula is  $\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}}$

where

$\bar{x}$  is mean ,  $\sum f$  is total frequency,  $f$ ,  $x$  each data value in turn.

$\sum f(x - \bar{x})^2$  means subtract the mean from each data value, square and multiply by  $f$  and finally add up the resulting values.

Since calculating the mean and standard deviation is tedious, we can save some of this work by extending columns of a frequency distribution table.

| $x$    | $f$ | $f \times x$ | $(x - \bar{x})^2$ | $f \times (x - \bar{x})^2$ |
|--------|-----|--------------|-------------------|----------------------------|
| .....  |     |              |                   |                            |
| .      |     |              |                   |                            |
| .      |     |              |                   |                            |
| .      |     |              |                   |                            |
| .      |     |              |                   |                            |
| Totals |     |              |                   |                            |

**EXAMPLE 1:** Kellan took a survey of 20 boys to find how many pairs of shoes each boy owned. He summarized the results in the table below:

| $x$   | $f$ |
|-------|-----|
| 1     | 1   |
| 2     | 4   |
| 3     | 9   |
| 4     | 6   |
| total | 20  |

Calculate the standard deviation.

**Answer:** set up a complete frequency table a fill in the columns

| $x$    | $f$ | $f \times x$ | $(x - \bar{x})^2$ | $f \times (x - \bar{x})^2$ |
|--------|-----|--------------|-------------------|----------------------------|
| 1      | 1   | 1            | $(1 - 3)^2 = 4$   | 4                          |
| 2      | 4   | 8            | $(2 - 3)^2 = 1$   | 4                          |
| 3      | 9   | 27           | $(3 - 3)^2 = 0$   | 0                          |
| 4      | 6   | 24           | $(4 - 3)^2 = 1$   | 6                          |
| totals | 20  | 60           |                   | 14                         |

$$\text{Mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{60}{20} = 3$$

$$\begin{aligned} \text{standard deviation } \sigma &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \sqrt{\frac{14}{20}} \\ &= 0.84 \end{aligned}$$

**EXAMPLE 2:** An exam conducted for 50 students resulted in the following marks:

|    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|
| 28 | 60 | 50 | 58 | 63 | 72 | 75 | 90 | 86 | 74 |
| 30 | 40 | 27 | 48 | 62 | 94 | 82 | 37 | 21 | 49 |
| 56 | 63 | 32 | 47 | 68 | 93 | 89 | 46 | 27 | 47 |
| 60 | 50 | 36 | 53 | 73 | 25 | 38 | 53 | 39 | 46 |
| 70 | 41 | 24 | 57 | 78 | 64 | 91 | 56 | 42 | 50 |

- i) Complete the table given below by categorizing the intervals as follows 20-29, 30-39, etc.

| Marks<br>(Class interval)<br>$x_1 \quad x_2$ | Tally | No. of<br>students<br>Freq ( $f$ ) | Cumulative<br>Frequency | Class<br>midpoint<br>( $x_m = \frac{x_1 + x_2}{2}$ ) | $f \cdot x_m$ | $(x - \bar{x})^2$ | $f \times (x - \bar{x})^2$ |
|----------------------------------------------|-------|------------------------------------|-------------------------|------------------------------------------------------|---------------|-------------------|----------------------------|
| 20 – 29                                      |       |                                    |                         |                                                      |               |                   |                            |
| 30 – 39                                      |       |                                    |                         |                                                      |               |                   |                            |
| 40 – 49                                      |       |                                    |                         |                                                      |               |                   |                            |
| 50 – 59                                      |       |                                    |                         |                                                      |               |                   |                            |
| 60 – 69                                      |       |                                    |                         |                                                      |               |                   |                            |
| 70 – 79                                      |       |                                    |                         |                                                      |               |                   |                            |
| 80 – 89                                      |       |                                    |                         |                                                      |               |                   |                            |
| 90 – 100                                     |       |                                    |                         |                                                      |               |                   |                            |
| Total ( $\Sigma$ )                           |       |                                    |                         |                                                      |               |                   |                            |

ii. Calculate the following:

- a) Mean  
b) Median interval  
c) Modal interval  
d) Standard deviation

Answers: It is a grouped data ➡

i.

| Marks<br>(Class interval)<br>$x_1 \quad x_2$ | Tally | No. of<br>students<br>Freq ( $f$ ) | Cumulative<br>Frequency | Class<br>midpoint<br>$x_m = \frac{x_1 + x_2}{2}$ | $f \cdot x_m$ | $(x - \bar{x})^2$ | $f \times (x - \bar{x})^2$ |
|----------------------------------------------|-------|------------------------------------|-------------------------|--------------------------------------------------|---------------|-------------------|----------------------------|
| 20 – 29                                      |       | 5                                  | 5                       | 24.5                                             | 122.5         | 1026.56           | 5132.8                     |
| 30 – 39                                      | I     | 6                                  | 11                      | 34.5                                             | 207           | 485.76            | 2914.56                    |
| 40 – 49                                      | <br>  | 9                                  | 20                      | 44.5                                             | 400.5         | 144.96            | 1304.64                    |
| 50 – 59                                      | <br>  | 9                                  | 29                      | 54.5                                             | 490.5         | 4.16              | 37.44                      |
| 60 – 69                                      | II    | 7                                  | 36                      | 64.5                                             | 451.5         | 63.36             | 443.52                     |
| 70 – 79                                      | II    | 7                                  | 43                      | 74.5                                             | 521.5         | 322.56            | 2257.92                    |
| 80 – 89                                      |       | 3                                  | 46                      | 84.5                                             | 253.5         | 781.76            | 2345.28                    |
| 90 – 100                                     |       | 4                                  | 50                      | 95                                               | 380           | 1479.17           | 5916.68                    |
| Total<br>( $\Sigma$ )                        |       | 50                                 |                         |                                                  | 2827          |                   | 20352.84                   |

ii.

a) Mean

$$\begin{aligned}\bar{x} &= \frac{\sum fx_m}{\sum f} \\ &= \frac{2827}{50} \\ &= 56.54\end{aligned}$$

b) Modal Interval

40 – 49 and 50 – 59

c) Median interval

$$\begin{aligned}&= \frac{\sum f}{2} \\ &= \frac{50}{2} \\ &= 25_{th} \text{ frequency} \\ &50 - 59\end{aligned}$$

d) Standard deviation

$$\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} = \sqrt{\frac{20352.84}{50}} = 20.18$$

## EFFECTS OF ADDING/MULTIPLYING BY A CONSTANT

**Note:** Let  $x_1, x_2, x_3, \dots, x_n$  be a set of values with mean  $\bar{x}$  and standard deviation  $\sigma$ .

1. If a same number  $k$  is added to each:  $x_1+k, x_2+k, x_3+k, \dots, x_n+k$  then new mean  $= \bar{x} + k$  while the standard deviation remains the same,  $\sigma$ .
2. If each number is multiplied by the same number  $k$ :  $kx_1, kx_2, kx_3, \dots, kx_n$  then new mean  $= k\bar{x}$  and new standard deviation  $= k\sigma$ .

### **Example**

A set of scores has a mean of 6 and standard deviation of 2. Find the new mean and standard deviation if

a) All of the scores are multiplied by 5

$$\begin{aligned}\text{Mean} &= 6 \times 5 & \text{s.d} &= 2 \times 5 \\ &= 30 & &= 10\end{aligned}$$

b) 20 is added to all of the scores

$$\begin{aligned}\text{Mean} &= 6 + 20 & \text{s.d} &= 2 \\ &= 26 & &\end{aligned}$$

- c) 2 is subtracted from all the scores

$$\begin{aligned}\text{Mean} &= 6 - 2 & \text{s.d} &= 2 \\ &= 4\end{aligned}$$

- d) All of the scores are divided by 2

$$\begin{aligned}\text{Mean} &= 6 \div 2 & \text{s.d} &= 2 \div 2 \\ &= 3 & &= 1\end{aligned}$$

#### EXERCISE 49:

- 2 sets of data are shown below: Set 1 = {1,3,5,7} and Set 2 = {5,15,25,35}. The mean and SD of Set 1 are given as  $\bar{x}$  and  $s$  respectively. Describe the mean and SD of the data in set 2 in terms of  $\bar{x}$  and  $s$ ?
- 2 sets of data are shown below:  
Set A = {a,b,c,d} and Set B = {a+2,b+2,c+2,d+2}.  
The mean and SD of Set A are given as  $\bar{x}$  and  $s$  respectively. Describe the mean and SD of the data in set B in terms of  $\bar{x}$  and  $s$ ?
- Complete the table shown below and find the Standard deviation.

| $x$   | $f$ | $fx$ | $(x - \bar{x})^2$ | $f(x - \bar{x})^2$ |
|-------|-----|------|-------------------|--------------------|
| 1     | 6   |      |                   |                    |
| 2     | 3   |      |                   |                    |
| 3     | 8   |      |                   |                    |
| 4     | 1   |      |                   |                    |
| 5     | 4   |      |                   |                    |
| total |     |      |                   |                    |

- The frequency table below represents the marks scored by a class in a test.

| Mark, $x$ | $f$ | $fx$ | $x - \bar{x}$ | $(x - \bar{x})^2$ | $f(x - \bar{x})^2$ |
|-----------|-----|------|---------------|-------------------|--------------------|
| 1         | 1   | 1    | -2            | 4                 | 4                  |
| 2         | 2   | 4    | -1            | 1                 | 2                  |
| 3         | 4   | 12   | 0             | 0                 | 0                  |
| 4         | 2   | 8    | 1             |                   |                    |
| 5         | 1   | 5    | 2             |                   |                    |
| Total     | 10  | 30   |               |                   |                    |

Complete the table and find the standard deviation of the distribution.

5. The table below shows the frequency distribution of the results when a child calls out the number 1, 2, 3, 4 and 5 randomly, 20 times.

|                   |   |   |   |   |   |
|-------------------|---|---|---|---|---|
| Number called out | 1 | 2 | 3 | 4 | 5 |
| Frequency         | 3 | 5 | 2 | 4 | 6 |

Find the:

- Mean of the distribution
- SD of the distribution

## **INTERPRETING INFORMATION FROM GRAPHS AND TABLES**

### **Exercise 50**

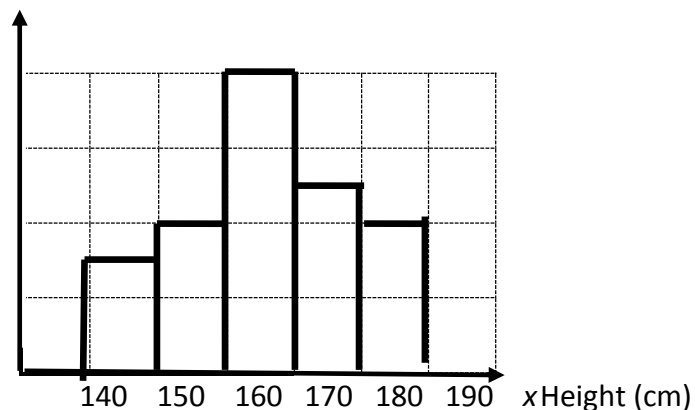
1. The heights of 15 science students, to the nearest cm, are given in the table below.

| Height (cm) | No. of students |
|-------------|-----------------|
| 161         | 1               |
| 162         | 4               |
| 163         | 5               |
| 164         | 2               |
| 165         | 3               |
| Total       | 15              |

Calculate the

- |                       |                   |                     |
|-----------------------|-------------------|---------------------|
| a) Mean               | b) Mode           | c) Range            |
| d) Median             | e) lower quartile | f) Inter – quartile |
| g) Standard deviation |                   |                     |

2. The heights of athletes in a team are given by the following histogram.

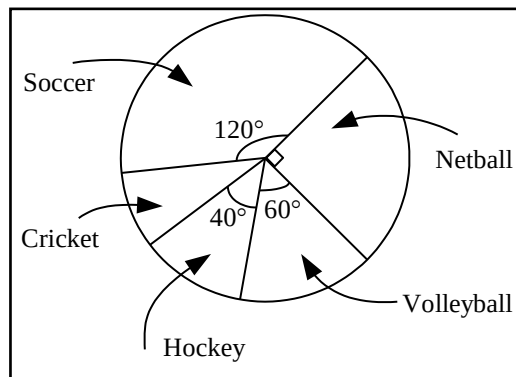


| Heights<br>Class<br>interval) | Freq<br>( $f$ ) | Cumulative<br>Frequency | Class<br>midpoint<br>( $x_m$ ) | $f \cdot x_m$ | $x - \bar{x}$ | $(x - \bar{x})^2$ | $f(x - \bar{x})^2$ |
|-------------------------------|-----------------|-------------------------|--------------------------------|---------------|---------------|-------------------|--------------------|
| 140 < 150                     |                 |                         |                                |               |               |                   |                    |
| 150 < 160                     |                 |                         |                                |               |               |                   |                    |
| 160 < 170                     |                 |                         |                                |               |               |                   |                    |
| 170 < 180                     |                 |                         |                                |               |               |                   |                    |
| 180 < 190                     |                 |                         |                                |               |               |                   |                    |
| Total<br>( $\Sigma$ )         |                 |                         |                                |               |               |                   |                    |

Copy and complete this table.

- Find mean and standard deviation
- Find the modal interval

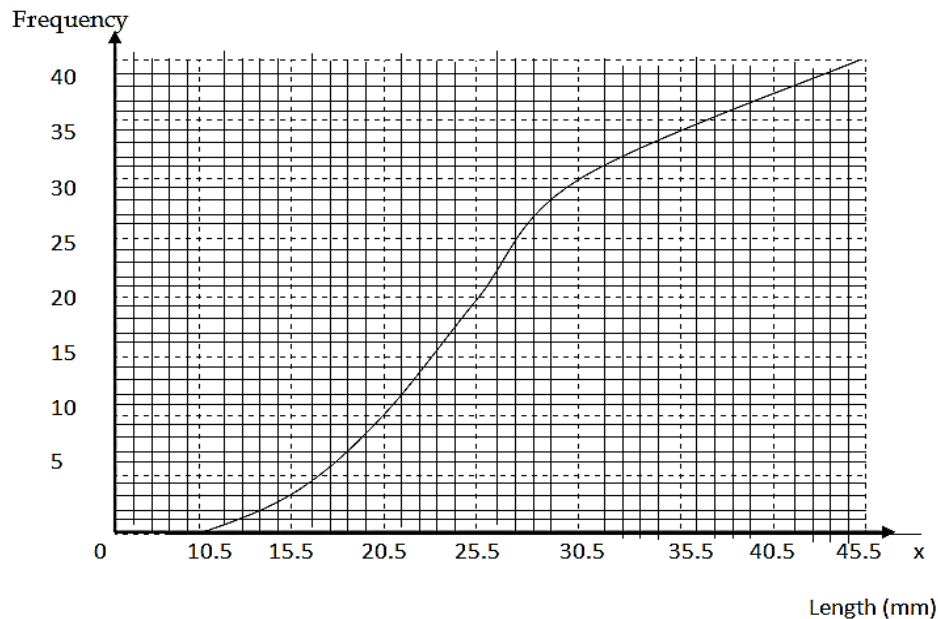
- The pie chart below shows the sports interests shown by a group of 200 students.



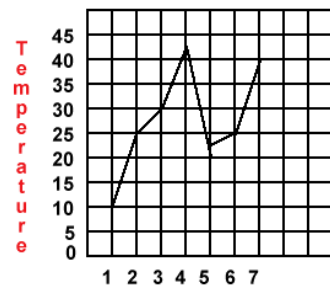
- What percentage of students like Netball?
- How many students like Cricket?
- How many students like Soccer?

4. A sample of leaves were collected from a flower garden and the lengths of the leaves were measured. The graph given below shows the results.

Cumulative Frequency Graph



- How many leaves were there in the sample?
  - What is the median?
  - What is the inter quartile range?
  - How many leaves have length 20.5mm or less?
  - How many leaves have length between 25.5 and 35.5mm?
  - How many leaves have lengths more than 15.5mm?
5. The line graph below shows the temperature recorded in January from 1<sup>st</sup> – 7<sup>th</sup>.



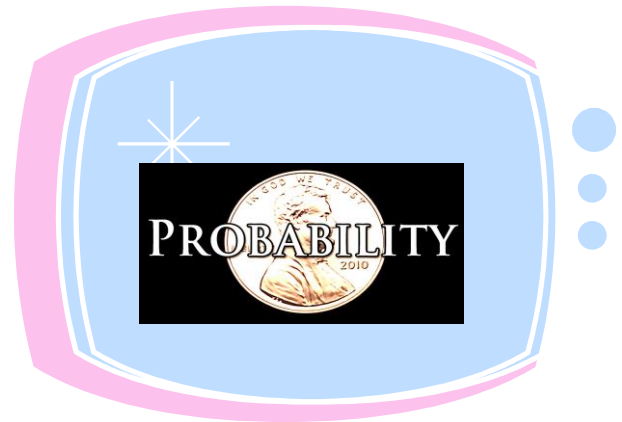
Average Daily Temperature for January 1-7 in Degrees Fahrenheit

- How much did the temperature rise from January 1st to the 4th?
- Over the 7 days, what trend do you notice?
- Which day had the greatest rise in temperature?
- Find the mean temperature and standard deviation of the seven days.
- What is the modal temperature recorded?

**Lesson of Life:** *The end and aim of all education is the development of character*

~ Francis W. Parker

# STRAND EIGHT



## SUB – STRAND 8.1

## Probability Experiments

Sampling with Replacement

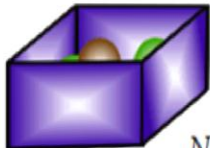


**Population**

This process is continued for as long as required

*Note the number remains the same*

Sampling without Replacement



**Population**

This process is continued for as long as required

*Number decreases since you are not replacing, that is not putting the item back*

## CONTENT LEARNING OUTCOMES

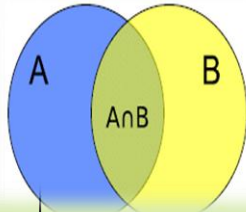
## Why is Learning Probability Important?

- Probability is used in gambling, sports, the gaming industry. It is used in weather forecasting, and in scientific research.
- The most **important** use of **probability** is to help us make decisions as we go through life!



- Introduction to probability
- Probability trees
- Probability with replacement
- Lattice Diagram
- Probability without replacement
- Venn diagrams
- Expected number

Venn diagrams are a popular visual method of representing sets and their relations and intersection.



## INTRODUCTION TO PROBABILITY

**Note:** Before we look at probability, the following definitions are useful:

- An activity where observations are recorded is called an **experiment**.
- Each time an experiment is performed, the performance is referred to as a **trial**.
- **Outcomes** are the result of an experiment. For example, tossing a coin gives either a Head or a Tail and rolling a die gives one of the numbers from 1 to 6.
- Each outcome is called a **sample point**.
- **Sample Space(S)** – set of all possible outcomes.  
Thus  $(S) = \{\text{Outcome1, Outcome 2, .....}, \text{Outcome n}\}$   
Example tossing a coin:  $S = \{\text{Head, Tail}\}$  or rolling a die:  $S = \{1, 2, 3, 4, 5, 6\}$
- **Event** is the subset of a sample space, for example getting a Head in tossing a coin.
- If an event **cannot occur** then its probability is 0. For instance the probability of getting a seven when you roll a die = 0 since it's impossible to get 7 [die has 6 sides only].
- If an event is **certain to occur** then its probability is 1.
- The **closer** the probability of an event is to **1**, the **more likely** it will occur.
- The probability of any event can range from **0 to 1**. i.e.  $0 \leq P(E) \leq 1$ .
- The sum of probabilities of all sample points in a sample space is equal to 1.  
 $P(S) = 1$  OR  $\sum \text{Probabilities} = 1$
- The **theoretical Probability of an event A** [The probability of event A is denoted by  $P(A)$ ].

$$P(A) = \frac{\text{Number of times event A occurs}}{\text{Number of possible outcomes}}$$

- The **experimental probability** of an event A happening is:

$$P(A) = \frac{\text{Number of times event A occurs}}{\text{Total number of trials}}$$

**Example 1:** If you roll a die 20 times and 9 of the rolls result in a “two”. What experimental probability is observed for the event “two”?

$$P(\text{getting two}) = \frac{9}{20} = 0.45$$

**Example 2:** What is the probability of getting an even number when throwing a die?



$$S = \{1, 2, 3, 4, 5, 6\}$$

$$P(\text{Even Number: } 2/4/6) = \frac{3}{6} = \frac{1}{2}$$

**EXERCISE 51:**

**[Multiple Choice Questions]**

1. A pack of cards consists of 7 red cards, 3 blue cards and 5 green cards. The pack is shuffled and a card is picked at random. The probability that a green card is picked is

A.  $\frac{1}{3}$

B. 5

C.  $\frac{5}{10}$

D.  $\frac{2}{3}$

2. A number is picked from the set  $N = \{1, 2, 3, 4, \dots, 20\}$ . The probability of selecting an even number and a multiple of 6 is

A.  $\frac{6}{20}$

B.  $\frac{17}{20}$

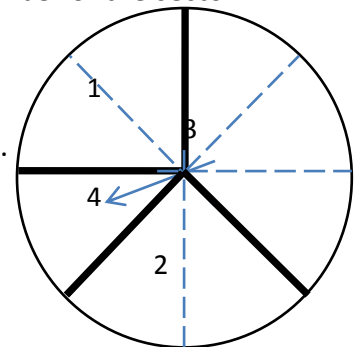
C.  $\frac{10}{20}$

D.  $\frac{3}{20}$

**[Questions requiring working]**

3. The 'pointer' of the spinner is spun. The outcome is the number of the sector on which the pointer stops.

- List the sample space.
- Are the outcomes on this spinner equally likely? Explain.
- Find the probability of getting an even number on the spinner.



- What is the probability of getting an odd number when throwing a die?
- A container contains 4 red, 2 green and 3 blue pins. What is the probability that a pin chosen at random is
  - green
  - not red
  - yellow

6. The heights of 15 students, to the nearest cm, are given in the table below.

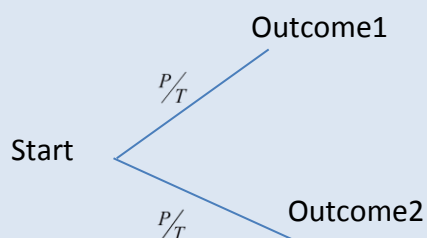
| Height (cm) | No. of students |
|-------------|-----------------|
| 151         | 1               |
| 152         | 4               |
| 153         | 5               |
| 154         | 2               |
| 155         | 3               |
| Total       | 15              |

What is the probability of selecting students with heights:

- Between 152 cm 154 cm
- Less than 153 cm
- At least 154cm

## PROBABILITY TREES

**Note:** To find probability of **two or more events**, you may use tree diagrams to list the Sample Space.



To calculate the probability for two branches: You multiply along the branches that give the required outcome.

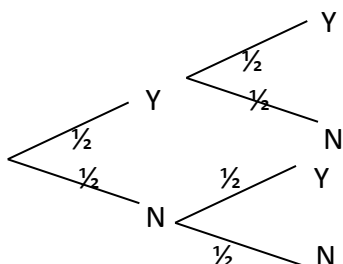
At each stage probabilities must add vertically to give 1.

**Example 1:** In an experiment, 2 men were asked whether they like watching soccer, to which their answers were either a Yes (Y) or No (N).

- Draw a probability tree.
- List the sample space for the experiment
- Find the probability that at least one of them like watching soccer.

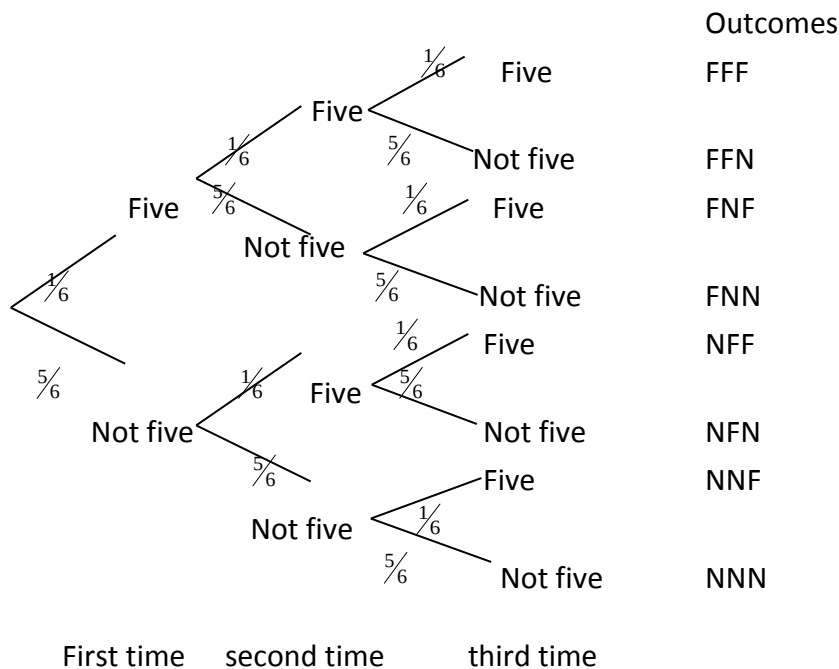
a) Tree Diagram

b)  $S = \{YY, YN, NY, NN\}$



$$\begin{aligned}
 P(\text{Minimum of } 1Y) &= P(YY) + P(YN) + P(NY) \\
 &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \\
 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}
 \end{aligned}$$

**EXAMPLE 2:** A die is rolled 3 times. Find the probability of getting exactly 2 five's.



$$\begin{aligned}
 P(\text{exactly 2 fives}) &= FFN + FNF + NFF \\
 &= \left(\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6}\right) + \left(\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}\right) + \left(\frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}\right) \\
 &= \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{5}{72}
 \end{aligned}$$

## PROBABILITY WITH REPLACEMENT

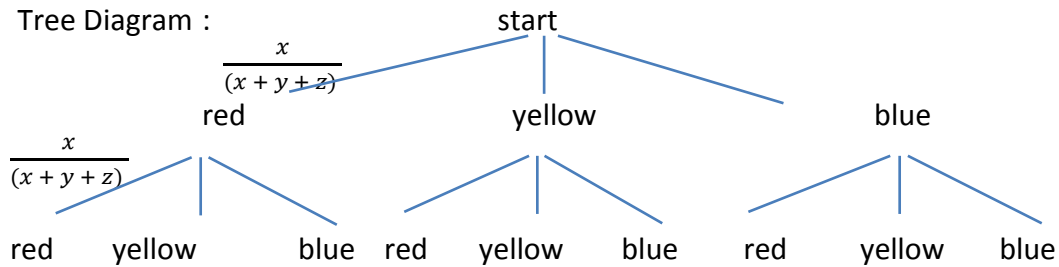
**Note:** Sampling **with replacement** is where you **put the object back after the first trial**

**EXAMPLE 3:** A box contains  $x$  red marbles,  $y$  yellow marbles &  $z$  blue marbles. Two marbles are drawn from the bag at random, with first being replaced before the 2<sup>nd</sup> is drawn.

What is the probability (in terms of  $x$ ,  $y$  &  $z$ ) that both marbles are red?

marbles  
 $S = \{ R, Y, B \}$   
 $x \ y \ z$  so total is  $(x + y + z)$

Tree Diagram :



$$P(\text{both red}) = RR = P(R) \times P(R) = \frac{\text{Part}}{\text{Total}} \times \frac{\text{Part}}{\text{Total}} = \frac{x}{(x + y + z)} \times \frac{x}{(x + y + z)} = \frac{x^2}{(x + y + z)^2}$$

## EXERCISE 52:

### [Multiple Choice Question]

1. A coin is tossed three times. The probability of getting the same outcome in the three tosses is

A.  $\frac{1}{4}$

B.  $\frac{1}{8}$

C.  $\frac{3}{8}$

D.  $\frac{7}{8}$

### [Questions requiring working]

2. A married couple plans to have 3 children. What is the probability that all three children will be of the same sex?
3. A box contains 2 green, 1 red and 2 yellow cubes of the same size. Two cubes are drawn with replacement, that is the first cube drawn is replaced before the second cube is drawn. What is the probability that:
  - a) Both cubes are red in color?
  - b) A yellow cube is drawn in the second draw?
4. Assuming that a woman is equally likely to give birth to a baby boy as to a baby girl, construct a sample space showing the different possibilities of bearing the next 3 children. Use B for boy and G for girl.

5. A fair six – sided die has 3 faces painted red, 2 faces painted yellow and 1 face painted blue. It is tossed twice.
  - a) Calculate the probability that both tosses give the same color.
  - b) Calculate the probability that the colors are different.
  
6. A box contains three balls (one red, one blue, one white). A ball is picked, its color is recorded, replaced then a second one is picked from the box.
  - a) List the Sample Space,  $S$ ?
  - c) What is the probability that you pick at least one blue?
  - b) What is the probability that you pick the same color?
  
7. Two coins, [one penny, one nickel] are tossed simultaneously, and the outcomes are recorded.
  - a) List the Sample Space,  $S$ ?
  - b) What is the number of sample points in  $S$ ,  $n(S)$ ?



## LATTICE DIAGRAM

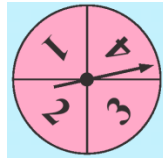
**Note: Lattice diagrams** are used to display the outcomes of two-stage events. This is much easier than tree diagrams.

|                |           | <b>Event 2</b> |           |       |           |
|----------------|-----------|----------------|-----------|-------|-----------|
|                |           | Outcome 1      | Outcome 2 | ..... | Outcome n |
| <b>Event 1</b> | Outcome 1 |                |           |       |           |
|                | Outcome 2 |                |           |       |           |
|                | ⋮         |                |           |       |           |
|                | Outcome n |                |           |       |           |

**EXAMPLE 1:** A spinner with numbers 1 to 4 is spun and an unbiased coin is tossed. Draw a lattice diagram and use it to give the following probabilities:

(a)  $P(\text{head and a 4})$

(b)  $P(\text{head or a 4})$



Spinner



Coin

**Answers:**

|   | 1   | 2   | 3   | 4   |
|---|-----|-----|-----|-----|
| H | H,1 | H,2 | H,3 | H,4 |
| T | T,1 | T,2 | T,3 | T,4 |

Total outcome = 8

$$P(\text{head and a 4}) = \frac{1}{8}$$

|   | 1   | 2   | 3   | 4   |
|---|-----|-----|-----|-----|
| H | H,1 | H,2 | H,3 | H,4 |
| T | T,1 | T,2 | T,3 | T,4 |

$$P(\text{head or a 4}) = \frac{5}{8}$$

**EXAMPLE 2:** A die is rolled and a coin is tossed simultaneously. Find the probability of getting a head and an odd number.



Die

Coin

|   | 1   | 2   | 3   | 4   | 5   | 6   |
|---|-----|-----|-----|-----|-----|-----|
| H | H,1 | H,2 | H,3 | H,4 | H,5 | H,6 |
| T | T,1 | T,2 | T,3 | T,4 | T,5 | T,6 |

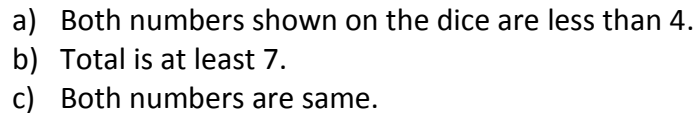
Total outcome = 12

$$P(\text{head and odd number}) = \frac{3}{12} = 0.25 \text{ or } \frac{1}{4}$$

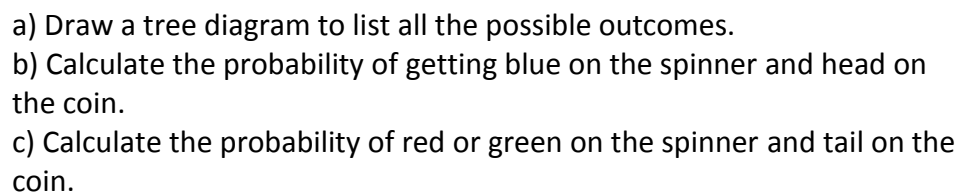
**[Multiple Choice Question]**

- A.      0.06
- B.      0.08
- C.      0.14
- D.      0.17

2) Two identical dice are rolled simultaneously. Draw a lattice diagram and find the following probabilities:



- red



208

## PROBABILITY WITHOUT REPLACEMENT

**Note:** Probability **without replacement** or dependent probability is where you **do not put the object back** so that the number is less than for the first draw. The probabilities for the 2 trials will **not** be the same, hence the tree diagram will be more appropriate to use.

**EXAMPLE 1:** A box contains 2 green, 1 blue and 2 yellow balls, all of the same shape and size. A ball is picked at random from the box. What is the probability that the ball picked will be:

- i) green?
- ii) not yellow?

If the first ball is picked and without replacing a second one is picked from the box, what is the probability that:

- iii) the first ball picked is blue and the second is yellow?
- iv) both balls are of the same color?

**Answers:**

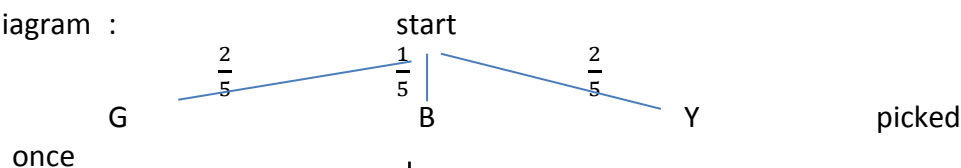


$$S = \{G, B, Y\}$$

$$2 \text{ } 1 \text{ } 2 \Rightarrow \text{Total is } 5$$

$$P(G) = \frac{2}{5}, P(B) = \frac{1}{5}, P(Y) = \frac{2}{5}$$

Tree Diagram :

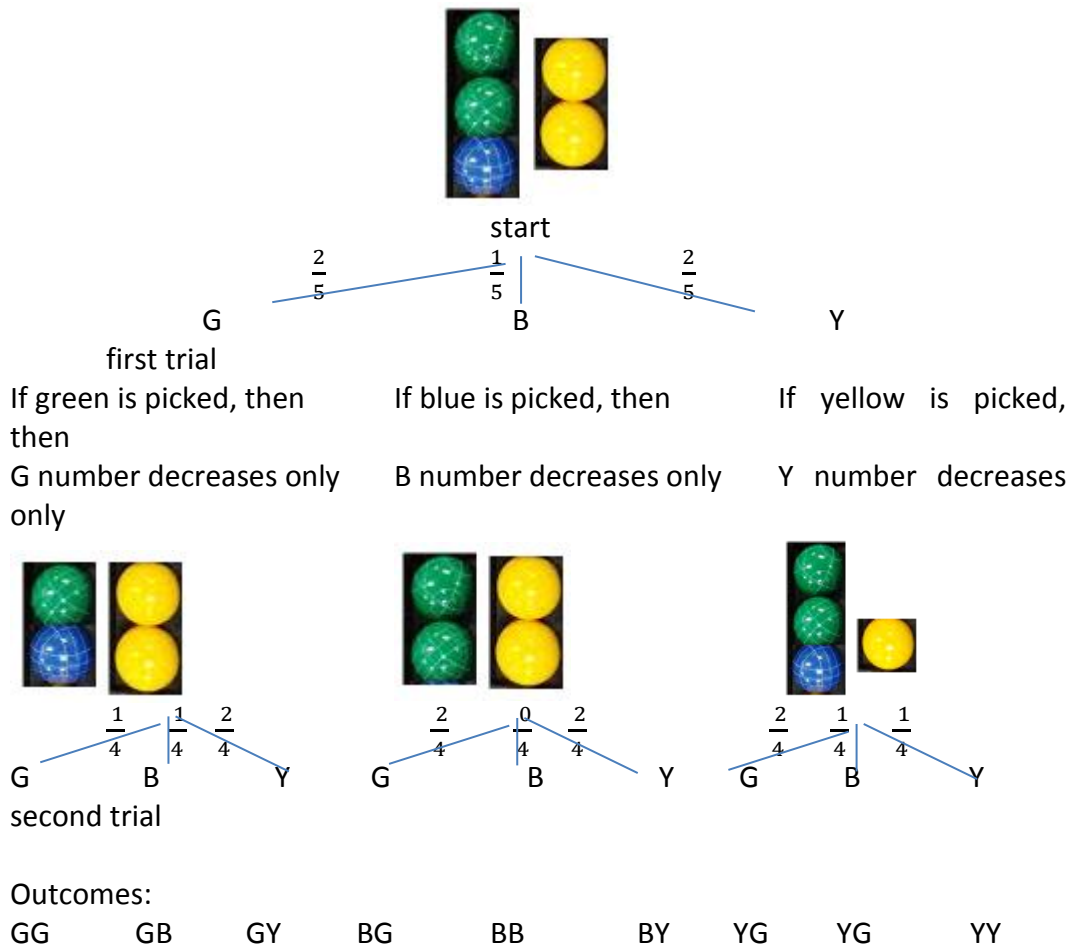


- i) green  
 $P(G) = \frac{2}{5}$  or 0.4

- ii) not yellow: do not include Y  
P(not Y) means  $P(B \& G) = \frac{2+1}{5} = \frac{3}{5}$  or 0.6  
Another method:  
 $1 - P(Y) = 1 - \frac{2}{5} = \frac{3}{5}$  or 0.6

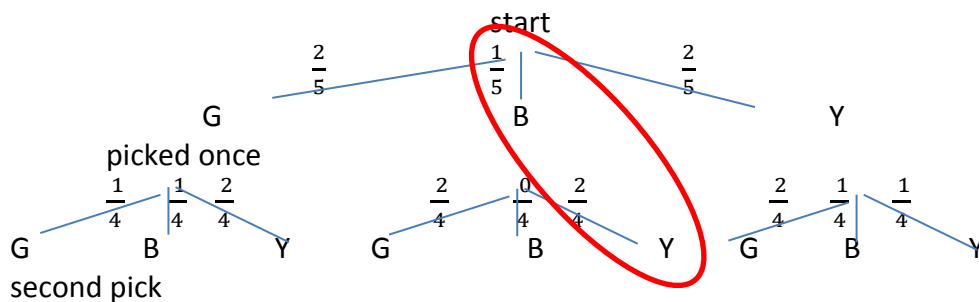
**Not replaced:** the number decreases to 4 since one is completely removed. Consider each scenario to get the probability.

Tree Diagram :

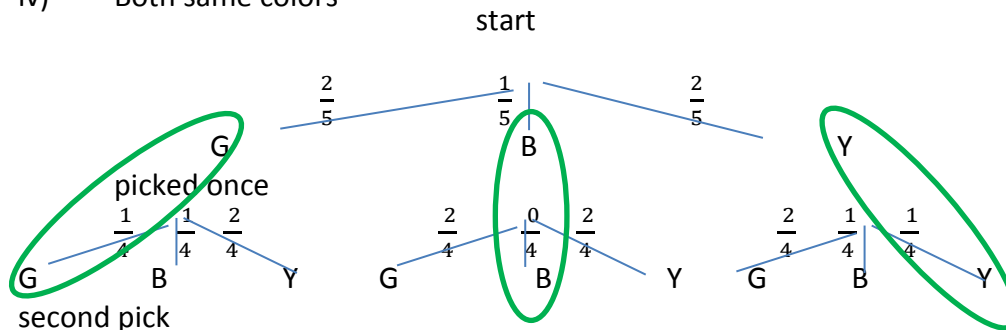


iii) the first is blue second is yellow

$$P(BY) = P(B) \times P(Y) = \frac{1}{5} \times \frac{2}{4} = \frac{1}{10} \text{ or } 0.1$$



iv) Both same colors



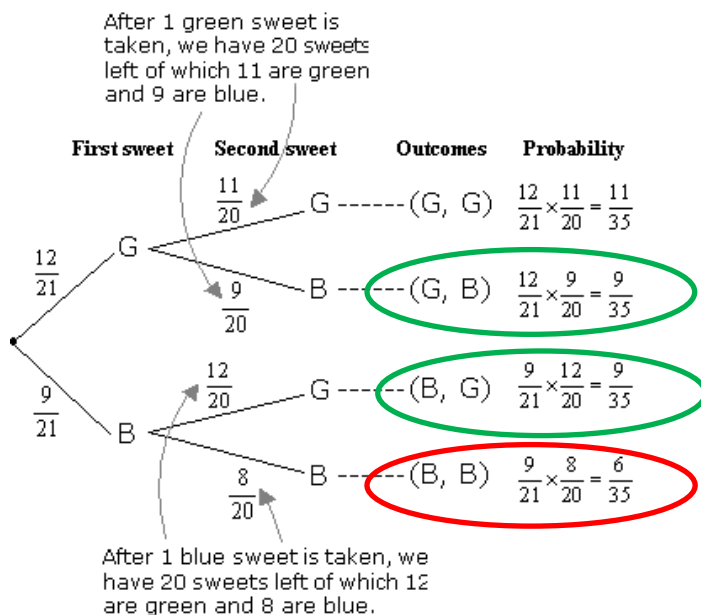
$$\begin{aligned}
 P(\text{same}) &= P(GG) + P(BB) + P(YY) \\
 &= P(G) \times P(G) + P(B) \times P(B) + P(Y) \times P(Y) \\
 &= \frac{2}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{0}{4} + \frac{2}{5} \times \frac{1}{4} \\
 &= \frac{1}{10} + 0 + \frac{1}{10} = \frac{1}{5} \text{ or } 0.2
 \end{aligned}$$

**Example 2:** A jar consists of 21 sweets. 12 are green and 9 are blue. Noa picked two sweets at random.

- Draw a tree diagram to represent the experiment.
- Find the probability that
  - both sweets are blue.
  - One sweet is blue and one sweet is green.

### Answers

a) Although both sweets were taken together it is similar to picking one sweet and then the second sweet without replacing the first sweet.



b)

|                                                                                                                                                                 |                                                                                                                                                                                                                                                                                                         |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>i) P(both sweets are blue)</p> <p><math>P(B, B)</math></p> $P(BB) = P(B) \times P(B)$ $= \frac{9}{21} \times \frac{8}{20} = \frac{6}{35} \text{ or } 0.1714$ | <p>ii) P(one sweet is blue and one is green)</p> <p><math>P(G, B) \text{ or } P(B, G)</math></p> $P(GB) + P(BG)$ $= P(G) \times P(B) + P(B) \times P(G)$ $= \frac{12}{21} \times \frac{9}{20} + \frac{9}{21} \times \frac{12}{20}$ $= \frac{12}{35} + \frac{12}{35} = \frac{24}{35} \text{ or } 0.5143$ |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

#### EXERCISE 54:

- Two marbles are picked from a jar containing 3 red, 4 blue and 5 green marbles. If a marble is picked without placing back, find the probability that out of the 2 marbles that are picked there is at least one green.
- Rita had a bag that contained 8 balls of the same size of which there were 3 red, 3 blue and 2 white balls. She picked a ball at random, noted its colour and picked another one without replacing the first ball. What is the probability that the balls picked were of the same colour?
- A box contains 2 green, 2 blue and 3 yellow crayons, all of the same shape and size. A crayon is picked at random from the box. What is the probability that the crayon picked will be:
  - yellow?
  - not green?



If the first crayon is picked and without replacing a second one is picked from the box, what is the probability that:

- the first crayon picked is yellow and the second is blue?
  - both crayons are green?
- A bag contains five red balls and three yellow balls. A ball is drawn at random and its color is noted. Without replacing it, a second ball is drawn and its color is also noted.
    - Draw a tree diagram to show the possible outcomes.
    - Find the probability that the second ball drawn is red.
    - What is the probability that both balls drawn have the same color?

5. A class of 25 Year 12 students has 10 girls and 15 boys. Three students were chosen at random to take part in a survey. Calculate the probability that all the students chosen were girls.

6. Metuisela has 12 coins in his pocket. Four are copper and eight are silver. He takes out two coins one after the other to lend it to his small brother Aron.

- (i) Draw a tree diagram to show the possible outcomes.
- (iii) Find the probability that both coins will be Silver.

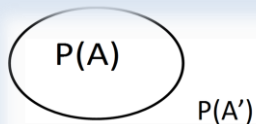


## VENN DIAGRAMS

**Note:** A **Venn diagram** is constructed with a collection of simple closed curves drawn in a plane as well as overlapping circles.

**Properties of Venn diagram:**

- The interior of the circle symbolically represents the elements of the set (A) while the exterior represents elements that are not members of the set ( $A'$  or  $A^c$ ). This is also referred to as **Compliment of A**.

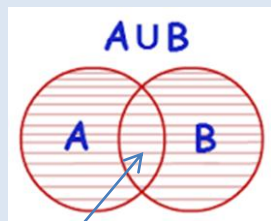


$$P(A) + P(A') = 1$$

$$P(A) = 1 - P(A^c)$$

$$P(A^c) = 1 - P(A)$$

- Empty Set** is a set with no elements common. Shown by  $\{ \}$  or  $\emptyset$ .  
 $A \cap B = \emptyset$  or  $\{ \}$  **null set**
- Two sets, A and B, is said to be **disjoint** if  $A \cap B = \emptyset$
- "Union"** of sets has the special symbol  $\cup$  that means it is the set containing all the elements in the first or in the second set.



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- $P(A \cap B)$ : "Intersection" of a set has special symbol  $\cap$  and it must be in both sets, A and B.

**EXAMPLE 1:** If set  $A = \{1, 3, 5, 7, 9\}$  and set  $B = \{2, 3, 5, 7, 10\}$ ,

- a) List  $A \cup B$ .
- b) List  $A \cap B$ .

**Answer:**

- a)  $A \cup B$ : join the two sets  
common is both  
 $A \cup B = \{1, 2, 3, 5, 7, 9, 10\}$

- b)  $A \cap B$ : the elements

$$A \cap B = \{3, 5, 7\}$$

**EXAMPLE 2:** In a school, the probability that a student can play volleyball is 0.46 and the probability that a student can play basketball is 0.34. The probability that the student can play **both** is 0.23.

Find the probability that a randomly selected student can play:

- a) Volleyball only.
- b) volleyball or basketball.
- c) Neither volleyball nor basketball.

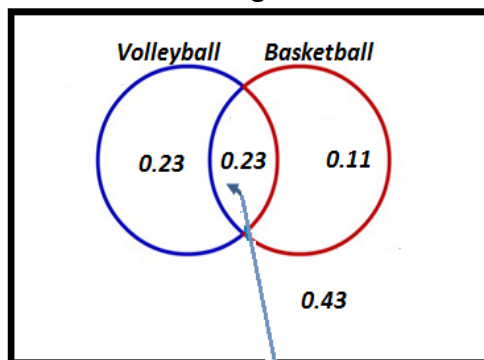


**Volleyball**



**Basketball**

Consider a Venn diagram:



*play Both (Volleyball  
and Basketball)*

*Total Probability to be  
one so subtract from **one**  
This is the compliment  
so 'neither nor'*

- a) Volleyball only = 0.23
- b) volleyball or basketball = Union so add the values of the two circles = 0.57  
or Use the formula:  
 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\begin{aligned}
 &P(\text{Volleyball or Basketball}) \\
 &= P(\text{Volleyball}) + P(\text{Basketball}) - P(\text{Volleyball and Basketball}) \\
 &= 0.46 + 0.34 - 0.23 \\
 &= 0.57
 \end{aligned}$$

- c) Neither volleyball nor basketball - outside the two overlapping circles  
 $outside = 1 - (0.23 + 0.23 + 0.11)$   
 $= 1 - 0.57 = 0.43$

**EXAMPLE 3:** Two sets, A and B, is **disjoint**. If  $P(A) = 0.4$   $P(B) = 0.3$ ,

- a) what is the probability of A and B occurring?  
 b) what is the probability of A **or** B occurring?

- |                                                                                     |                     |                                                                                                                                      |
|-------------------------------------------------------------------------------------|---------------------|--------------------------------------------------------------------------------------------------------------------------------------|
| <p>a) The word <b>and</b> refers to <math>P(A \cap B)</math>:</p> $P(A \cap B) = 0$ | <br> <br> <br> <br> | <p>b) The word <b>or</b> refers to <math>P(A \cup B)</math>:</p> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $= 0.4 + 0.3 - 0$ $= 0.7$ |
|-------------------------------------------------------------------------------------|---------------------|--------------------------------------------------------------------------------------------------------------------------------------|

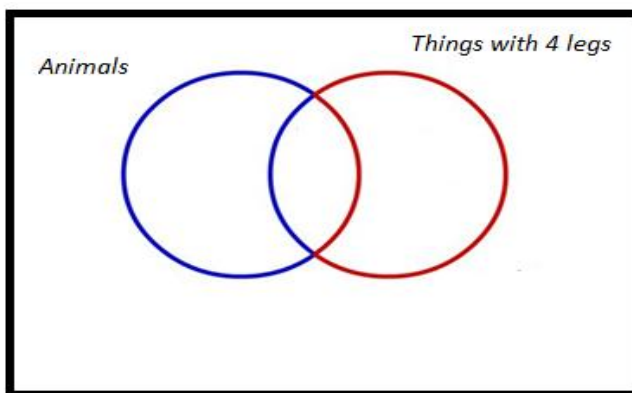
**EXERCISE 55:**

- The probability of Tom passing Statistics is 0.74, while the probability that he passes Calculus is 0.65. The probability that he fails both is 0.18. What is the probability that he passes both.
- $P(C \cup D) = 0.7$ ,  $P(C) = 0.3$ ,  $P(D) = 0.5$ . Calculate  $P(C \cap D)$ .
- Given the probabilities  $P(A) = \frac{2}{5}$ ,  $P(B) = \frac{1}{4}$ ,  $P(A \cap B) = \frac{1}{10}$ ; draw a Venn diagram and use it to evaluate:
  - $P(A \cup B)$
  - $P(A')$
  - $P(B')$
- In a class, the probability that a student has a handkerchief is 0.54 that a student has a home – made lunch is 0.68. The probability that a student has both a handkerchief a home – made lunch is 0.25. What is the probability that a student selected at rom will have either a handkerchief or a home – made lunch?
- In a Suva market, the probability that a vendor has driven a car last year is 0.78, the probability that the vendor has driven a truck is 0.15 and the probability that the vendor has driven both vehicles is 0.06. Find the probability that a randomly selected vendor has
  - Driven a car only.
  - Driven 1 type of the vehicles but not the other.
  - Not driven any type of the vehicles last year.

6. The PEMAC teacher in a secondary school knows that the probability of a Year 9 students entering the athletics sport is 0.47, the swimming sport is 0.23 and no sports is 0.51. What is the probability that a year 9 student enters both sports?
7. In Savusavu town, the probability that a household has a TV set is 0.73 and the probability that a household has a computer is 0.42. The probability that a household has a TV and a computer is 0.18. Find the probability that a randomly selected household has:
- A TV set only
  - Neither a TV set nor a computer

**Activity:**

Place the following in the appropriate place in your Venn diagram.



**Lesson of Life:** *Mathematics is the supreme judge; from its decisions there is no appeal* ~Tobias Dantzig

## EXPECTED NUMBER

**Note:** Recall that probability is the chance that an event will happen. We can use this probability to predict the number of times an event will happen in the future. This is known as Expected value. To find the Expected number, multiply the probability of the outcome by the number of trials.

$$E = n \times p$$

where

E – Expected number

n – Number of trials

p – Probability of event

**EXAMPLE 1:** A biased coin, with probability of heads = 0.42, is tossed 300 times. What is the expected number of heads that will be obtained?

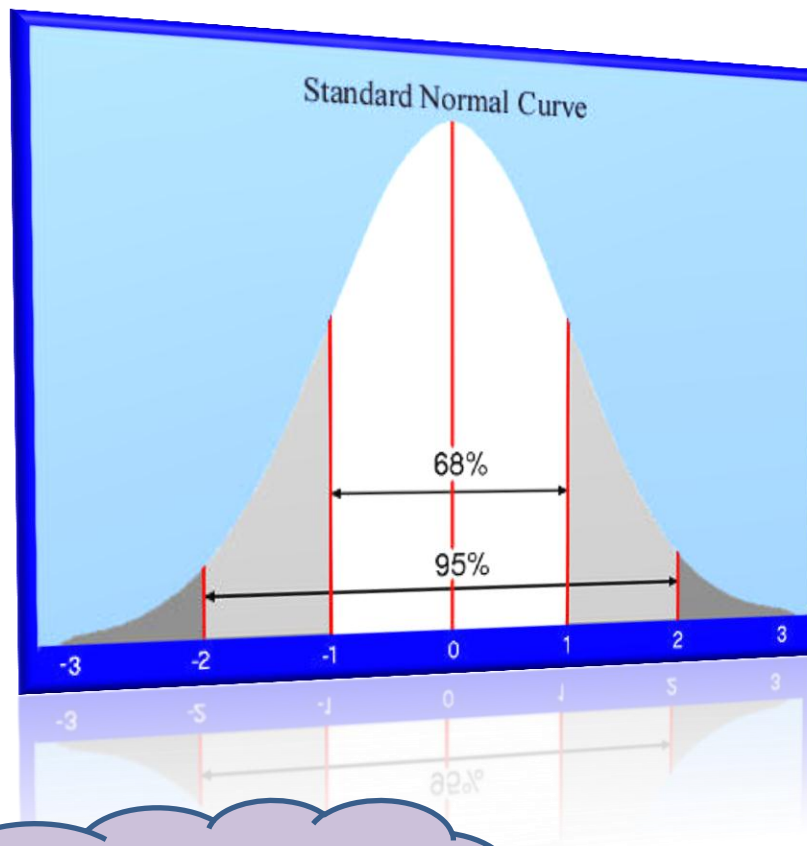
[Biased is where the outcomes are not equally likely]

$$\begin{aligned} E &= n \times p \\ &= 300 \times 0.42 = 126 \end{aligned}$$

### **EXERCISE 56:**

1. An uneven dice with eight equally likely outcomes is tossed. The probability of getting “three” is 0.2.
  - a) Write this probability as a percentage.
  - b) If you toss this dice 30 times, how many times would you expect to get “three”?
2. A math teacher said that the probability of a surprise test on any day this year is 0.15.
  - a) If there are 180 school days in the year, how many surprise tests can you expect to have?
  - b) Is it certain that you will have those many tests? Explain.
3. A box contains 2 red balls and a yellow ball. The probability of picking a red ball is  $\frac{2}{3}$ .
  - a) What is the probability of picking a yellow ball?
  - b) If you try to pick 12 times, how many times would you expect it to be yellow?
  - c) How many times would you expect it to be red if you try 36 times?
4. Alisi is a shooter for her school’s netball team. In today’s game, she has made 12 goals out of 20 attempts.
  - a) What is the probability of the successful shot of the day?
  - b) How many of her next 30 shots, would you expect her to make goals?





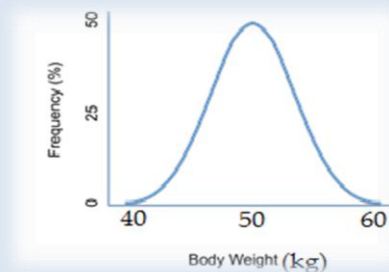
**CONTENT LEARNING  
OUTCOMES**

- Characteristics of a normal curve and related terms
- Standardizing
- Finding probabilities using normal distribution tables
- Application of normal distribution

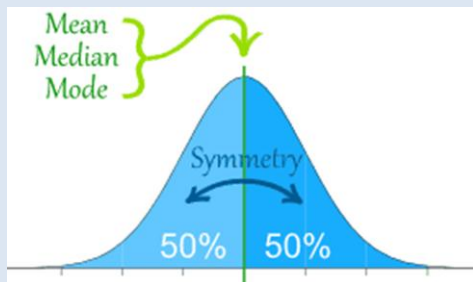


## CHARACTERISTICS OF A NORMAL DISTRIBUTION

**Note:** Suppose we measure each student's weight in a large secondary school and then summarize the findings with a graph that displays different body weights on the horizontal axis (the x -axis) and the frequency (% of students) of each weight on the vertical axis (the y - axis), we would get a graph similar to the one shown below:



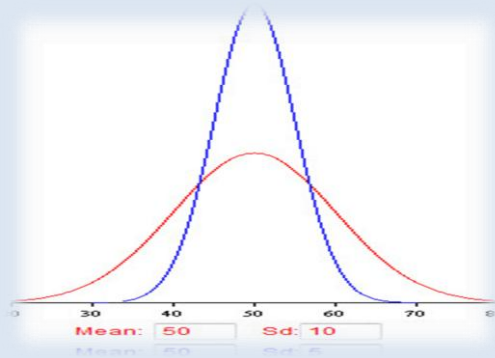
There are several noteworthy characteristics of this graph. It is bell-shaped with a single peak in the center, and it is symmetrical. If the distribution is perfectly symmetrical with a single peak in the center, then the mean value, the mode, and the median will be all the same. Many variables have similar characteristics, which are characteristic of so-called **normal distributions**.



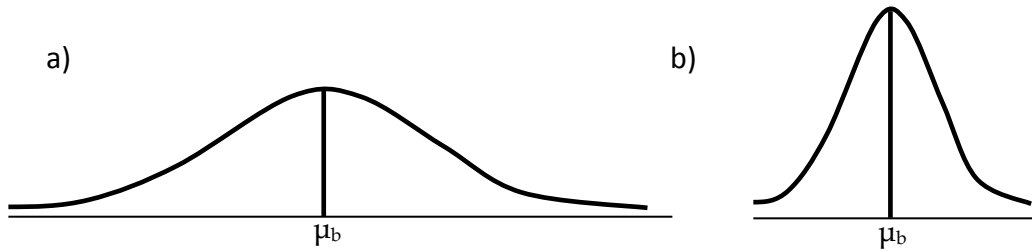
The **Normal Distribution** has the following features:

mean = median = mode  
symmetrical about the center  
50% of values are less than the mean  
while 50% are greater than the mean

- When the standard deviation is large, the curve is short and wide; when the standard deviation is small, the curve is tall and narrow.



**EXAMPLE 1:** Which of the following normal probability distribution has the smallest standard deviation?

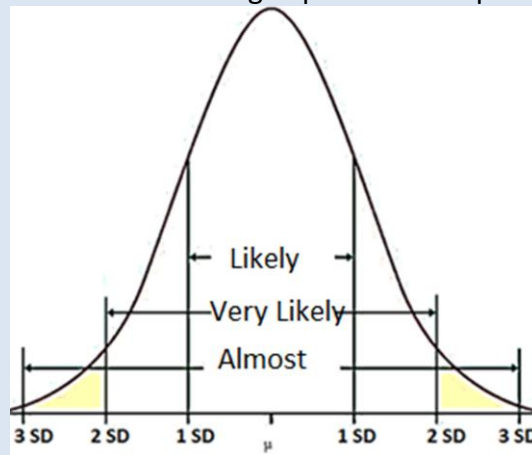
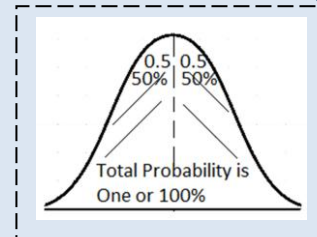


Answer: ( b )

## NORMAL DISTRIBUTIONS TERMS

**Note:** The normal distribution is a continuous probability distribution. This has several implications for probability.

- The total **area under** the normal curve is equal to **one**.
- A score is
  - **probable** or **likely** to lie in this range :  $\mu - 1\sigma < x < \mu + 1\sigma$
  - **very likely** or **very probably** to lie in this range :  $\mu - 2\sigma < x < \mu + 2\sigma$
  - **almost certain** to lie in this range :  $\mu - 3\sigma < x < \mu + 3\sigma$



**EXAMPLE 2:** Your Company packages sugar in one kg bags. When you weigh a sample of bags you get these results:

1007g, 1032g, 1002g, 983g, 1004g, 1040g, 1021g, 999g, 1009g,  $x$  g

- Find the mean weight of 9 bags of sugar.
- Another weight  $x$  g is to be included such that the mean weight is 1010g. Find the weight of the 10<sup>th</sup> bag.
- Work out the standard deviation.
- Assume that the weight of the sugar is normally distributed with a mean of 1010g standard deviation of 16g. Draw the normal distribution curve.
- Between what weights is the bag likely to be?
- Between what weights is the bag almost certain to be?

**Answers:**

$$a) \bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{1007 + 1032 + \dots + 1009}{9} = \frac{9097}{9} = 1010\frac{7}{9}g$$

- b) Another weight is added so  $n$  is now 10 .

$$\bar{x} = \frac{\text{Total}}{n}$$

$$1010 = \frac{\text{Total} + x}{10}$$

$$1010 \times 10 = 9097 + x$$

$$x = 10100 - 9097$$

$$\therefore x = 1003g$$

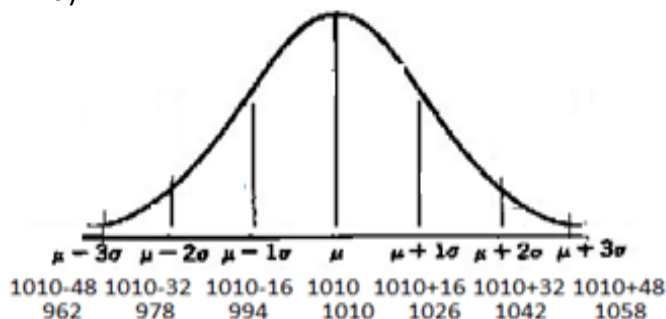
- c) Put calculator mode on SD. Press each number with M+ sign.

Press

|       |    |      |    |     |    |    |    |
|-------|----|------|----|-----|----|----|----|
| 100   | M+ | 1032 | M+ | 100 | M+ | 98 | M+ |
| 100   | M+ | 104  | M+ | 102 | M+ | 99 | M+ |
| 100   | M+ | 100  | M+ |     |    |    |    |
| Shift | 2  | 2    | =  |     |    |    |    |

$$\sigma = 15.86 \approx 16g$$

- d)



e) **likely** to be:

$$\mu - 1\sigma < x < \mu + 1\sigma$$

$$994g < x < 1026g$$

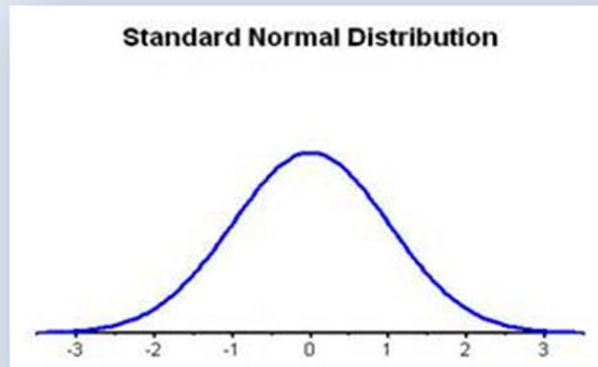
f) **almost certainly**:

$$\mu - 3\sigma < x < \mu + 3\sigma$$

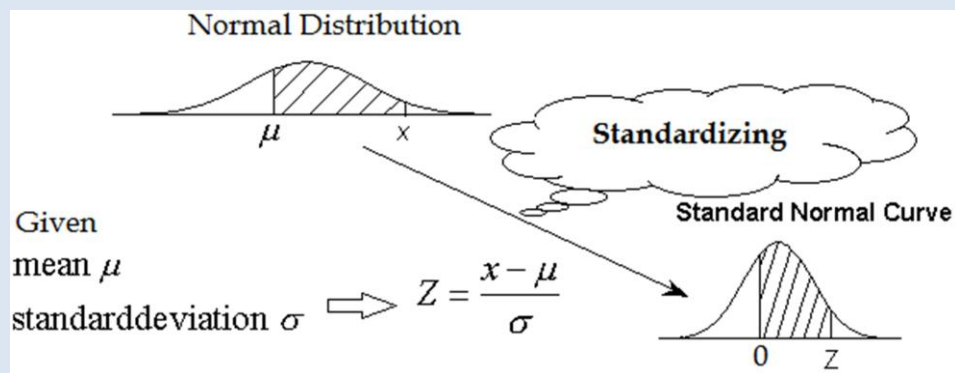
$$962g < x < 1058g$$

## STANDARD NORMAL DISTRIBUTION

**Note:** Each set of data that is normally distributed has a different mean and standard deviation. It is therefore impossible to give probabilities for every situation. Instead, we convert to standard normal distribution.



It is a normal distribution with mean of 0 standard deviation of 1.



- To convert a value to a Standard Score:
  - first subtract the mean,
  - then divide by the Standard Deviation
- For negative z – score  
⇒ use the positive equivalent as the normal curve is a **symmetrical graph**.
- For clarity purpose, draw and shade the required region.

$$Z = \frac{x - \mu}{\sigma}$$

**EXAMPLE 3:** For the following scores, convert it to the z – score and draw the required region.

a) mean  $\mu = 250$   
Standard Deviation  $\sigma = 25$   
 $x = 230$   
 $P(x > 230)$

b) mean  $\mu = 63$   
Standard Deviation  $\sigma = 5$   
 $x = 55$   
 $P(x < 55)$

c) mean  $\mu = 135$   
Standard Deviation  $\sigma = 4$   
 $x = 140$   
 $P(x > 140)$

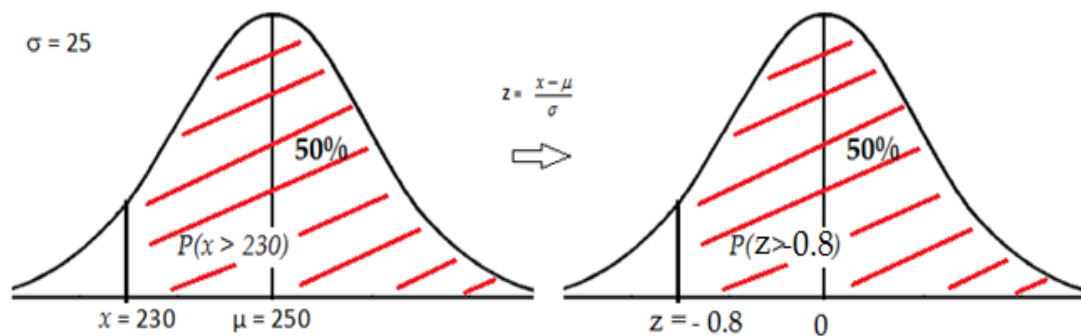
d) mean  $\mu = 10.3$   
Standard Deviation  $\sigma = 0.8$   
 $x = 11.5$   
 $P(x < 11.5)$

e) mean  $\mu = 62$   
Standard Deviation  $\sigma = 15$   
 $x = 25, \quad x = 77$   
 $P(25 < x < 77)$

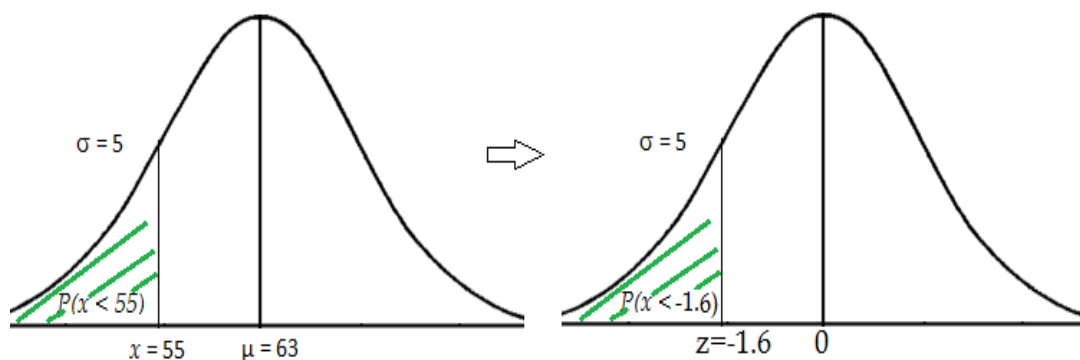
Answer:

a)

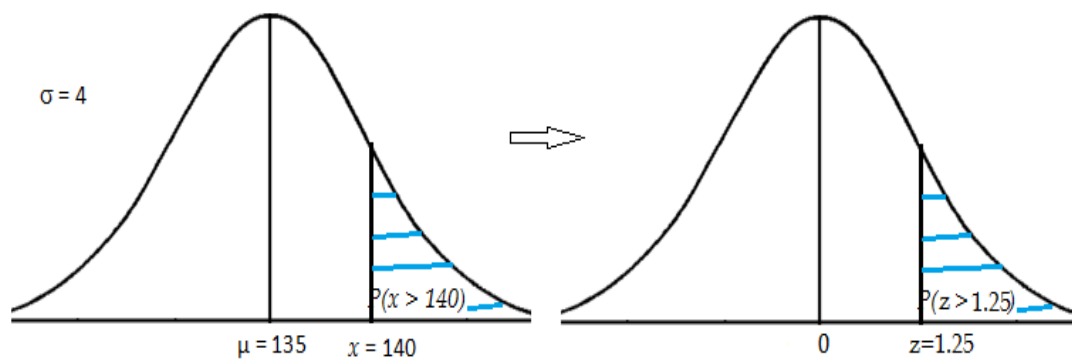
$$Z = \frac{x - \mu}{\sigma} = \frac{230 - 250}{25} = -0.8$$



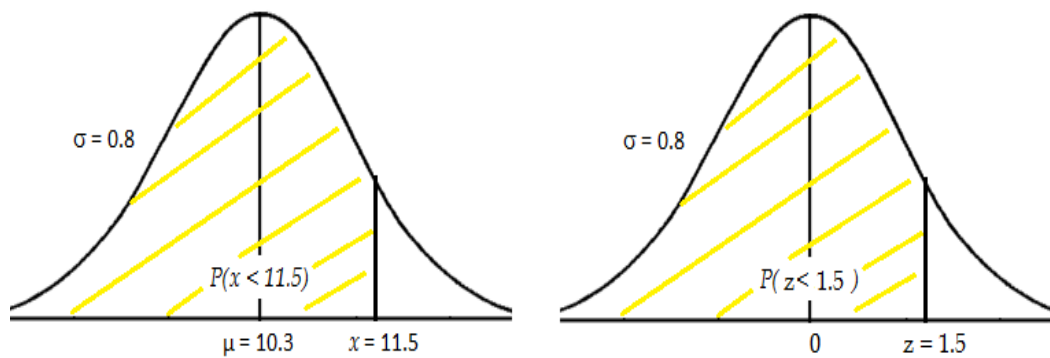
b)  $Z = \frac{x - \mu}{\sigma} = \frac{55 - 63}{5} = -1.6$



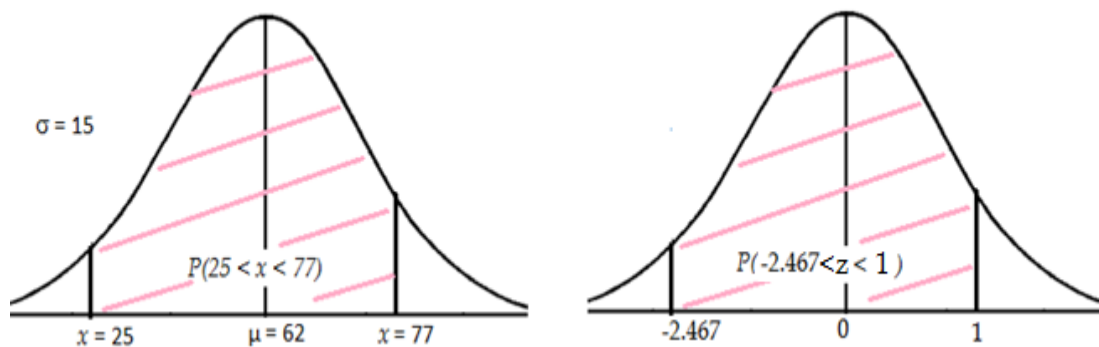
$$c) Z = \frac{x - \mu}{\sigma} = \frac{140 - 135}{4} = 1.25$$



$$a) Z = \frac{x - \mu}{\sigma} = \frac{11.5 - 10.3}{0.5} = 1.5$$



$$b) Z = \frac{x - \mu}{\sigma} = \frac{25 - 62}{15} = -2.467, \quad Z = \frac{x - \mu}{\sigma} = \frac{77 - 62}{15} = 1$$



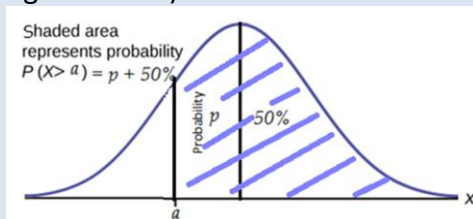
## FINDING PROBABILITIES

**Note:** The probability that a normal random variable  $X$  equals any particular value is 0.

- Less than  $a$  (option "Up to  $a$ ")
- Greater than  $a$  (option " $a$  onwards")

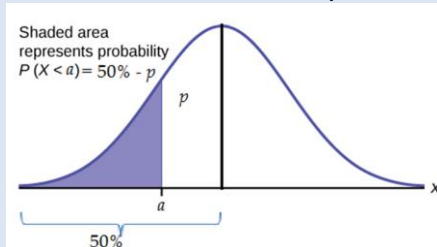
Consider the probabilities of different regions:

- The probability that  $x$  is **greater than  $a$**  on the left side is equal to the area under the normal curve bounded by  **$a$  plus 0.5** (as indicated by the non-shaded area in the figure below).



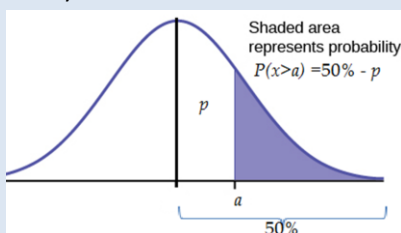
$$P(x > a) = 0.5 + p$$

- The probability that  $x$  is **less than  $a$**  on the left side is equal to the area under the normal curve bounded by  **$a$  subtracted from 0.5**. (as indicated by the shaded area).



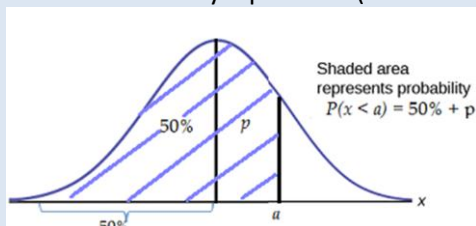
$$P(x < a) = 0.5 - p$$

- The probability that  $x$  is **greater than  $a$**  on the right side is equal to the area under the normal curve bounded by  **$a$  subtracted from 0.5**. (as indicated by the shaded area).



$$P(x < a) = 0.5 - p$$

- The probability that  $x$  is **less than  $a$**  on the right side is equal to the area under the normal curve bounded by  **$a$  plus 0.5**. (as indicated by the non-shaded area).



$$P(x > a) = 0.5 + p$$

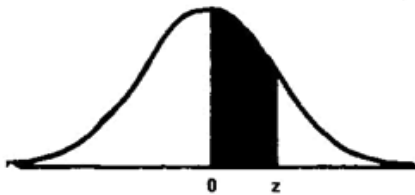
## TABLES FOR THE STANDARD NORMAL DISTRIBUTION

**Note:** To find the probability of standard scores,  $z$  – value is not enough. You need to know how to use the table.

### Steps

- Find the  $z$  – score with reference to question.
- Draw and shade the required region. } Done previously
- Use table to find the probability in relation to  $z$  –score
- Add 0.5 or Subtract from 0.5 or subtract from 1 etc. depending on shaded area.

### AREAS UNDER NORMAL PROBABILITY CURVE



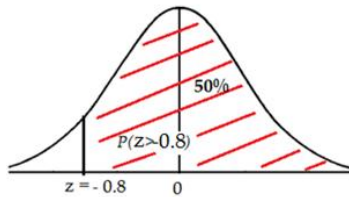
The tabulated value is the probability that the standardized normal variate  $Z$  (with  $\mu=0$ ,  $\sigma=1$ ) lies between 0 and  $z$ .

e.g.  $P(0 < Z < 1.43) = 42.36\%$

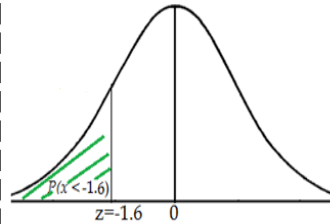
| $z$ | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     |  | 1 | 2 | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|--|---|---|----|----|----|----|----|----|----|
| 0.0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 |  | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0754 |  | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 |  | 4 | 8 | 12 | 15 | 19 | 22 | 27 | 31 | 35 |
| 0.3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 |  | 4 | 8 | 11 | 15 | 19 | 22 | 26 | 30 | 34 |
| 0.4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 |  | 4 | 7 | 11 | 14 | 18 | 22 | 25 | 29 | 32 |
| 0.5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 |  | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 27 | 31 |
| 0.6 | .2258 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2518 | .2549 |  | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 0.7 | .2580 | .2612 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 |  | 3 | 6 | 9  | 12 | 15 | 18 | 21 | 24 | 27 |
| 0.8 | .2881 | .2910 | .2939 | .2967 | .2996 | .3023 | .3051 | .3078 | .3106 | .3133 |  | 3 | 6 | 8  | 11 | 14 | 17 | 19 | 22 | 25 |
| 0.9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 |  | 3 | 5 | 8  | 10 | 13 | 15 | 18 | 20 | 23 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 |  | 2 | 5 | 7  | 9  | 12 | 14 | 16 | 18 | 21 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 |  | 2 | 4 | 6  | 8  | 10 | 12 | 14 | 16 | 19 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 |  | 2 | 4 | 5  | 7  | 9  | 11 | 13 | 15 | 16 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 |  | 2 | 3 | 5  | 6  | 8  | 10 | 11 | 13 | 14 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 |  | 1 | 3 | 4  | 6  | 7  | 8  | 10 | 11 | 13 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 |  | 1 | 2 | 4  | 5  | 6  | 7  | 8  | 10 | 11 |
| 1.6 | .4452 | .4463 | .4474 | .4484 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 |  | 1 | 2 | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| 1.7 | .4554 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 |  | 1 | 2 | 3  | 3  | 4  | 5  | 6  | 7  | 8  |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4699 | .4706 |  | 1 | 1 | 2  | 3  | 4  | 4  | 5  | 6  | 6  |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4756 | .4761 | .4767 |  | 1 | 1 | 2  | 2  | 3  | 4  | 4  | 5  | 5  |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 |  | 0 | 1 | 1  | 2  | 2  | 3  | 3  | 4  | 4  |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 |  | 0 | 1 | 1  | 2  | 2  | 2  | 3  | 3  | 4  |
| 2.2 | .4861 | .4864 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 |  | 0 | 1 | 1  | 1  | 2  | 2  | 2  | 3  | 3  |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 |  | 0 | 0 | 1  | 1  | 1  | 2  | 2  | 2  | 2  |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 |  | 0 | 0 | 1  | 1  | 1  | 1  | 1  | 2  | 2  |
| 2.5 | .4938 | .4940 | .4941 | .4943 | .4945 | .4946 | .4948 | .4949 | .4951 | .4952 |  | 0 | 0 | 0  | 1  | 1  | 1  | 1  | 1  | 1  |
| 2.6 | .4953 | .4955 | .4956 | .4957 | .4959 | .4960 | .4961 | .4962 | .4963 | .4964 |  | 0 | 0 | 0  | 0  | 1  | 1  | 1  | 1  | 1  |
| 2.7 | .4965 | .4966 | .4967 | .4968 | .4969 | .4970 | .4971 | .4972 | .4973 | .4974 |  | 0 | 0 | 0  | 0  | 0  | 1  | 1  | 1  | 1  |
| 2.8 | .4974 | .4975 | .4976 | .4977 | .4977 | .4978 | .4979 | .4979 | .4980 | .4981 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 1  |
| 2.9 | .4981 | .4982 | .4982 | .4983 | .4984 | .4984 | .4985 | .4985 | .4986 | .4986 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 1  |
| 3.0 | .4987 | .4987 | .4987 | .4988 | .4988 | .4989 | .4989 | .4989 | .4990 | .4990 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.1 | .4990 | .4991 | .4991 | .4991 | .4992 | .4992 | .4992 | .4992 | .4993 | .4993 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.2 | .4993 | .4993 | .4994 | .4994 | .4994 | .4994 | .4994 | .4994 | .4995 | .4995 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.3 | .4995 | .4995 | .4995 | .4996 | .4996 | .4996 | .4996 | .4996 | .4996 | .4997 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.4 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4997 | .4998 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.5 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 | .4998 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.6 | .4998 | .4998 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.7 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.8 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 | .4999 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 3.9 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 | .5000 |  | 0 | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |

**EXAMPLE 4:** For the previous scores, find the probability

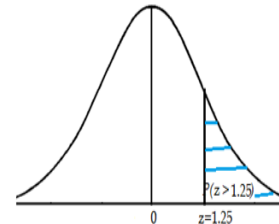
a)  $P(x > 230) = P(z > -0.8)$   
 $= 0.2881 + 0.5$   
 $= 0.7881$  or 78.81%



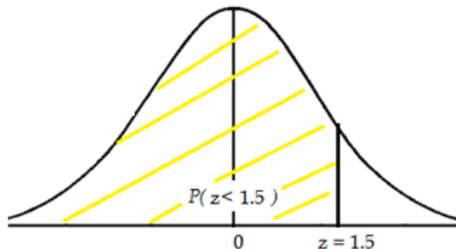
b)  $P(x < 55) = P(z < -1.6)$   
 $= 0.5 - 0.4452$   
 $= 0.0548$  or 5.48%



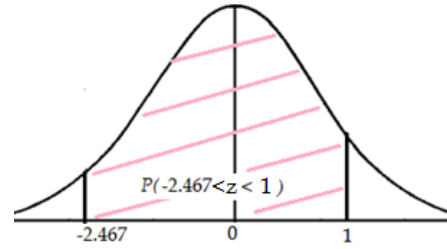
c)  $P(x > 140) = P(z > 1.25)$   
 $= 0.5 - 0.3944$   
 $= 0.1056$  or 10.56%



d)  $P(x < 11.5) = P(z < 1.5)$   
 $= 0.5 + 0.4332$   
 $= 0.9332$  or 93.32%



e)  $P(25 < x < 77) = P(-2.467 < z < 1)$   
 $= 0.4932 + 0.3413$   
 $= 0.8345$  or 83.45%



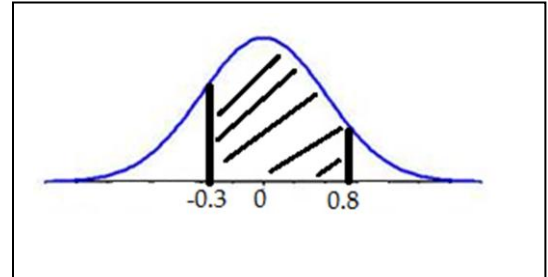
| z   | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 1 | 2 | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---|---|----|----|----|----|----|----|----|
| 0.0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0754 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 | 4 | 8 | 12 | 15 | 19 | 22 | 27 | 31 | 35 |
| 0.3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 | 4 | 8 | 11 | 15 | 19 | 22 | 26 | 30 | 34 |
| 0.4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 | 4 | 7 | 11 | 14 | 18 | 22 | 25 | 29 | 32 |
| 0.5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 27 | 31 |
| 0.6 | .2258 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2518 | .2549 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 0.7 | .2580 | .2612 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 | 3 | 6 | 9  | 12 | 15 | 18 | 21 | 24 | 27 |
| 0.8 | .2881 | .2910 | .2939 | .2967 | .2996 | .3023 | .3051 | .3078 | .3106 | .3133 | 3 | 6 | 8  | 11 | 14 | 17 | 19 | 22 | 25 |
| 0.9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 | 3 | 5 | 8  | 10 | 13 | 15 | 18 | 20 | 23 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 | 2 | 5 | 7  | 9  | 12 | 14 | 16 | 18 | 21 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 | 2 | 4 | 6  | 8  | 10 | 12 | 14 | 16 | 19 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 | 2 | 4 | 5  | 7  | 9  | 11 | 13 | 15 | 16 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 | 2 | 3 | 5  | 6  | 8  | 10 | 11 | 13 | 14 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 | 1 | 3 | 4  | 6  | 7  | 8  | 10 | 11 | 13 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 | 1 | 2 | 4  | 5  | 6  | 7  | 8  | 10 | 11 |
| 1.6 | .4452 | .4463 | .4474 | .4484 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 | 1 | 2 | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| 1.7 | .4554 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 | 1 | 2 | 3  | 3  | 4  | 5  | 6  | 7  | 8  |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4699 | .4706 | 1 | 1 | 2  | 3  | 4  | 4  | 5  | 6  | 6  |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4756 | .4761 | .4767 | 1 | 1 | 2  | 2  | 3  | 4  | 4  | 5  | 5  |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 | 0 | 1 | 1  | 2  | 2  | 3  | 3  | 4  | 4  |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 | 0 | 1 | 1  | 2  | 2  | 2  | 3  | 3  | 4  |
| 2.2 | .4861 | .4864 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 | 0 | 1 | 1  | 1  | 2  | 2  | 2  | 3  | 3  |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 | 0 | 0 | 1  | 1  | 1  | 2  | 2  | 2  | 2  |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 | 0 | 0 | 1  | 1  | 1  | 1  | 1  | 2  | 2  |

4931  
+ 1  
= 4932

### EXAMPLE 5

The amount of weight lost by people on a special diet is normally distributed with a mean of 8 kg and standard deviation of 3.5 kg. If 1500 people were on this diet, how many are expected to lose between 6.95 kg and 10.8 kg?

$$\begin{aligned}P(6.95 < x < 10.8) &= P\left(\frac{6.95 - 8}{3.5} < Z < \frac{10.8 - 8}{3.5}\right) \\&= P(-0.3 < Z < 0.8) \\&= 0.1179 + 0.2881 \\&= 0.406\end{aligned}$$



$$\begin{aligned}\text{Expected number} &= 0.406 \times 1500 \\&= 609\end{aligned}$$

### EXERCISE 57:

#### [Multiple Choice Question]

1. Every time people use liquid soap dispenser, the volume of soap produced is normally distributed with a mean of 7 mL and standard deviation of 1.5 mL. The probability that over 9mL of soap is dispensed is

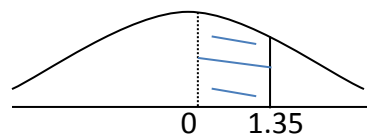
- |           |           |
|-----------|-----------|
| A. 40.82% | B. 9.13%  |
| C. 9.18%  | D. 40.87% |

#### [Questions requiring working]

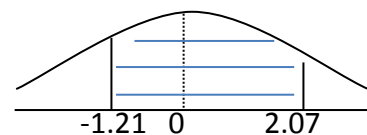
2. IQ scores from a test are normally distributed with a mean of 100 standard deviation of 16.5. If 100 students sit the IQ test, how many will score between 67 and 133?

3. Use the tables for the standard normal distribution to find the probabilities represented by shaded regions:

(a)



(b)



4. An average light bulb manufactured by the Acme Corporation lasts 300 days with a standard deviation of 50 days. Assuming that bulb life is normally distributed, what is the probability that an Acme light bulb will last less than 365 days?

5. Suppose scores on an IQ test are normally distributed. If the test has a mean of 100 and a standard deviation of 10, what is the probability that a person who takes the test will score between 90 and 110?
6. A soft-drink dispensing machine is regulated so that it discharges an average of 200 mL per cup. The amount of soft drink is normally distributed with a standard deviation of 15 mL.
  - a) What is the probability that a cup contains less than 191 mL of soft drink?
  - b) What is the probability that a cup contains between 191 mL and 209 mL of soft drink?
  - c) How many cups of soft drink will overflow if 230 mL cups are used for the next 1000 discharges?
7. The final exam marks of a class of 500 students are normally distributed with a mean of 62 marks and standard deviation of 15 marks.  
A student is selected at random. What is the probability that the student has scored a mark between 62 and 80?
8. The weights of brick chosen to pave a driveway are normally distributed with a mean of 1.4 kg and a standard deviation of 80g.
  - a) Between which 2 weights will a brick chosen at random probably lie?
  - b) What is the probability that a brick chosen at random will weigh less than 1.50 kg?
  - c) The number of bricks required to pave the driveway is 1260. Estimate the numbers that weigh between 1.45kg and 1.55kg.
9. A vegetable farmer in the Western Division supplies a large number of capsicums also known as 'bell pepper' to a national supermarket chain. The weight of the capsicums is normally distributed, with mean 185g and standard deviation 30g.
  - a) What is the probability that a capsicum chosen at random weighs more than 170g?
  - b) What is the probability that a capsicum chosen at random weighs between 215g and 230g?
  - c) Estimate the number of capsicums in a consignment of 600 that could be expected to weigh over 250g.

**Lesson of Life:** *Life is a Math Equation. In order to gain the most, you have to know how to convert the negatives into positives ~ Anonymous*

# STRAND NINE

# CALCULUS

## SUB – STRAND 9.1

## Limits



Actual Screenshot



When does a limit exist?

The limit of my attraction to you as it goes to infinity, it does not exist....  
 $\lim_{x \rightarrow \infty} \text{my attraction to you} = \text{does not exist}$

heart  
sigh



## CONTENT LEARNING OUTCOMES

- Definition of limits
- Limits from algebraic expressions

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

## DEFINITION OF LIMITS

Consider  $f(x) = \frac{1-x^2}{1-x}$ . Let's find  $f(1)$

$$\begin{aligned} f(1) &= \frac{1-1^2}{1-1} \\ &= \frac{0}{0} \end{aligned}$$

Any number **divided by zero is undefined**.

Let's see what happens when we take values close to 1 and substitute it in place

of  $x$  in the expression  $\frac{1-x^2}{1-x}$

| $x$  | $\frac{1-x^2}{1-x}$ |
|------|---------------------|
| 0.97 | 1.97                |
| 0.98 | 1.98                |
| 0.99 | 1.99                |
| 1    | undefined           |
| 1.01 | 2.01                |
| 1.02 | 2.02                |
| 1.03 | 2.03                |



The values get closer and closer to 2 as  $x$  approaches 1.



It turns out that as  $x$  approaches 1,  $f(x)$  approaches 2.

In standard mathematical notation, we say **the limit (in short lim) of  $f(x)$  as  $x$  approaches 1 is 2**. This relationship would be described as

$$\begin{aligned} \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{1-x^2}{1-x} \\ &= 2 \end{aligned}$$

which means "as  $x$  approaches 1, the value of  $f(x)$  approaches 2".

To make it simple, the limit of a function is what the function "approaches" (  $y$  value) when the input (the variable " $x$ ") approaches a specific value.

Thus **the limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$**  is written as

$$\lim_{x \rightarrow a} f(x) = L$$

## LIMITS FROM ALGEBRAIC EXPRESSIONS

Step 1: Directly substitute in the expression. If upon substitution you get

- $\frac{a}{b}$  then this is the limit.
- $\frac{0}{b}$  then the limit is 0.
- $\frac{a}{0}$  then the limit does not exist.
- $\frac{0}{0}$  then the limit may exist, go to step 2.

Step 2: Simplify by some algebraic method

**EXAMPLE 1:** Evaluate

a)  $\lim_{x \rightarrow -3} x^2 - 3$

Direct  
substitute  
the value -3,  
$$\begin{aligned}\lim_{x \rightarrow -3} x^2 - 3 \\&= (-3)^2 - 3 \\&= 9 - 3 \\&= 6\end{aligned}$$

b)  $\lim_{x \rightarrow 1} (2x + 1)(x - 1)$

Direct substitute  
the value 1,  
$$\begin{aligned}\lim_{x \rightarrow 1} (2x + 1)(x - 1) \\&= (2 \times 1 + 1)(1 - 1) \\&= 3 \times 0 \\&= 0\end{aligned}$$

**EXAMPLE 2:** Evaluate  $\lim_{x \rightarrow 2} \frac{x^2 + 4x + 3}{x - 2}$

Directly substitute the value 2,  
$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2 + 4x + 3}{x - 2} \\&= \frac{x^2 + 4x + 3}{x - 2} = \frac{2^2 + 4(2) + 3}{2 - 2} \\&= \frac{4 + 8 + 3}{0} = \frac{15}{0}\end{aligned}$$

In this case, there is no limit or  
limit does not exist.

**EXAMPLE 3:** Evaluate

a)  $\lim_{x \rightarrow 2} \frac{2-x}{4-x^2}$

Directly substitute the value 2,

$$\lim_{x \rightarrow 2} \frac{2-x}{4-x^2} = \frac{2-2}{4-4} = \frac{0}{0}$$

So factorize, simplify then substitute 4

$$\lim_{x \rightarrow 2} \frac{2-x}{4-x^2}$$

$$\frac{2-x}{4-x^2} = \frac{2-x}{(2-x)(2+x)}$$

$$\lim_{x \rightarrow 2} \frac{1}{2+x} = \frac{1}{2+2} = \frac{1}{4}$$

b)  $\lim_{x \rightarrow -2} \frac{16-4x^2}{x+2}$

Directly substitute the value - 2,

$$\lim_{x \rightarrow -2} \frac{16-4x^2}{x+2} = \frac{16-4(-2)^2}{-2+2} = \frac{0}{0}$$

So factorize, simplify then substitute -2

$$\lim_{x \rightarrow -2} \frac{16-4x^2}{x+2} \quad 4 \text{ is common so}$$

factorize the numerator

$$\frac{4\left(\frac{16}{4} - 4x^2\right)}{x+2} = \frac{4(4-x^2)}{x+2} = \frac{4(2^2-x^2)}{x+2}$$

Use difference of squares:

$$a^2 - b^2 = (a-b)(a+b)$$

$$\frac{4(2^2-x^2)}{x+2} = \frac{4(2-x)(2+x)}{x+2} = 4(2-x)$$

$$\therefore \lim_{x \rightarrow -2} 4(2-x) = 4(2-(-2)) = 16$$

**EXERCISE 61:** Evaluate the following limits:

a)  $\lim_{x \rightarrow -5} 5x$

b)  $\lim_{x \rightarrow 10} x-10$

c)  $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$

d)  $\lim_{x \rightarrow 2} \frac{x+5}{x-2}$

e)  $\lim_{x \rightarrow -1} \frac{3x^2-3}{x+1}$

f)  $\lim_{x \rightarrow 3} \frac{2x^2-5x-3}{x-3}$

g)  $\lim_{x \rightarrow 1} (x^2+5x-3)$

h)  $\lim_{x \rightarrow 3} \frac{3x-5}{x-3}$

i)  $\lim_{x \rightarrow 2} \frac{4x^2-2x-1}{x^2-2}$

j)  $\lim_{x \rightarrow 0} \frac{x^3-7x}{5}$

k)  $\lim_{x \rightarrow 0} \frac{(1+x)^2-1}{x}$

l)  $\lim_{x \rightarrow 2} \frac{3x^2-x-10}{x^2-4}$

m)  $\lim_{x \rightarrow -2} \frac{x^2-4}{x^2-x-6}$

n)  $\lim_{x \rightarrow 4} \frac{x^2-2x-8}{x-4}$

o)  $\lim_{x \rightarrow 0} \frac{5x^2+x}{x}$

**Lesson of Life:** Mathematics is like music. When the band plays in tune, it feels right; when the horns play at, it feels wrong. If you're doing mathematics the right way, it just feels right. It makes sense ~ John Mosley

## Basic rules of differentiation

$y = f(x) = c \rightarrow f'(x) = 0$ , where  $c$  is any constant.

$y = f(x) = x^n \rightarrow f'(x) = n x^{n-1}$ , where  $n$  is any real number.

$y = f(x) = c u(x) \rightarrow f'(x) = c u'(x)$

$y = f(x) = u(x) \pm v(x) \rightarrow f'(x) = u'(x) \pm v'(x)$

Recall that

$$f'(x) \quad y' \quad \frac{dy}{dx} \quad D_x f(x)$$

all represent the derivative. All rules follow from the definition of the derivative.

**CONTENT LEARNING  
OUTCOMES**

- Definition of the Derivative Function
- The rule for finding derivatives
- Derivative of Fractional and Negative Indices
- Higher derivatives
- Applications

## DEFINITION OF THE DERIVATIVE FUNCTION

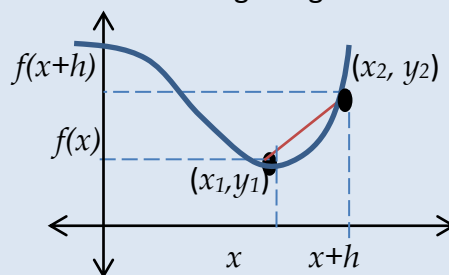
**Note:** The derivative is an important tool in calculus that represents **change** in a **function with respect to** one of its **variables**. The process of finding derivatives is often called **differentiation**.

- Given a function ' $f(x)$ ' or ' $y$ ', there are many ways to denote the **derivative** of ' $f$ ' with respect to ' $x$ '. The most common ways are  $\frac{df}{dx}$  and  $f'(x)$  or  $y'$ .

Keep in mind that  $\frac{df}{dx}$  is not a fraction and does not mean  $df \div dx$ .

For second order derivatives, the notation  $\frac{d^2y}{dx^2}$ ,  $f''(x)$  or  $y''$  is often used.

- Definition** Consider a diagram given below:



The gradient of the line passing through the two points is:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}$$

Now as  $h \rightarrow 0$ , the chord becomes **tangent** at point  $(x_1, y_1)$ .

Thus the gradient of tangent at  $(x_1, y_1) = \lim_{h \rightarrow 0} (\text{gradient of chord})$

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= f'(x)$$

This is known as '**Differentiation from the first principles**'

Steps to be used in finding the derivative using the First Principles of Differentiation:

- First find  $f(x+h)$  by substituting  $x+h$  for every place of  $x$  in  $f(x)$
- Compute  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  and simplify.
- Find limit by substituting 0 in place of  $h$ . If you get  $\frac{0}{0}$  you need to simplify before substituting.

**EXAMPLE 1:** Differentiate  $f(x) = 3x - 2$  by using the **First Principles**

$$\begin{aligned} f(x) &= 3x - 2 \\ f(x+h) &= 3(x+h) - 2 \\ f(x+h) &= 3x + 3h - 2 \end{aligned}$$

- Compute

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{3x + 3h - 2 - (3x - 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x + 3h - \cancel{2} - 3x + \cancel{2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h}{h} \\ &= \lim_{h \rightarrow 0} 3 \\ &= 3 \\ \therefore f'(x) &= 3 \end{aligned}$$

**EXAMPLE 2:** Find the derivative of  $f(x) = 5x^2 + 2x$  by using the **First Principles**.

$$\begin{aligned} f(x+h) &= 5(x+h)^2 + 2(x+h) \\ &= 5(x^2 + 2xh + h^2) + 2x + 2h \\ &= 5x^2 + 10xh + 5h^2 + 2x + 2h \end{aligned}$$

$$\begin{aligned} f(x+h) - f(x) &= (5x^2 + 10xh + 5h^2 + 2x + 2h) - (5x^2 + 2x) \\ &= 10xh + 5h^2 + 2h \end{aligned}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{10xh + 5h^2 + 2h}{h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} 10x + 5h + 2 \\ &= 10x + 2 \end{aligned}$$

$$\begin{aligned} \frac{10xh + 5h^2 + 2h}{h} &= \frac{h(10x + 5h + 2)}{h} \\ \text{(Cancel } h) & \\ &= 10x + 5h + 2 \end{aligned}$$

## THE RULE FOR FINDING DERIVATIVES

**Note:** To differentiate,  $y = kx^n$ , multiply the coefficient  $k$  by the power  $n$  and then reduce the power by one.

$$y = x^n$$

$$y' \text{ or } \frac{dy}{dx} = n \cdot x^{n-1}$$

$$f(x) = ax^n$$

$$f'(x) = (n \times a)x^{n-1}$$

Example. Prove that the derivative of a **constant**,  $k$ , is **zero**.

$$y = k$$

$$\Rightarrow y = kx^0 \text{ recall } x^0 = 1$$

$$y' = (0 \times k)x^{0-1} \\ = 0$$

**Example 2:** Differentiate the following

a)  $f(x) = 5$

Derivative  
of a  
constant is  
zero,  
 $f'(x) = 0$

b)  $y = 4x + 6$

Recall  $x = x^1$   
 $y = 4x^1 + 6$   
*Constant*  
 $y' = (1 \times 4)x^{1-1} + 0$   
 Recall  $x^0 = 1$   
 $\therefore y' = 4$

c)  $f(x) = 2x^3 - 3x + 1$

$f(x) = 2x^3 - 3x + 1$   
*Constant*  
 $f'(x) = 2x^3 - 3x^1 + 1$   
 $f'(x) = (3 \times 2)x^{3-1} - (1 \times 3)x^{1-1} + 0$   
 $f'(x) = 6x^2 - 3x^0 + 0$   
 $\therefore f'(x) = 6x^2 - 3$

d)  $g(x) = (3 - 2x)^2$

*Expand and simplify*  
 $g(x) = (3 - 2x)^2$   
 $g(x) = 9 - 12x + 4x^2$   
 $g'(x) = -12 + 8x$

## DERIVATIVE OF FRACTIONAL AND NEGATIVE INDICES

**Note:** The law of indices is required to change in the form of base and power before applying differentiation of power rule.

- Negative indices

$$\frac{1}{x} = x^{-1}$$

$$\frac{x}{x^n} = x^{-n}$$

- Rational indices

$$\sqrt{x} = x^{1/2}$$

$$\sqrt[n]{x^m} = x^{m/n}$$

**Example 3:** Differentiate the following

a)  $f(a) = \frac{5}{a} + 2$

$$f(a) = 5a^{-1} + 2$$

$$f'(a) = (-1 \times 5)a^{-1-1} + 0$$

$$= -5a^{-2}$$

$$\therefore f'(a) = -5a^{-2} \text{ or } -\frac{5}{a^2}$$

b)  $m = \sqrt{n} - n^2$

$$m = n^{1/2} - n^2$$

$$\frac{dm}{dn} = \frac{1}{2}n^{1/2-1} - 2n^{2-1}$$

$$= \frac{1}{2}n^{-1/2} - 2n$$

$$\therefore \frac{dm}{dn} = \frac{1}{2}n^{-1/2} - 2n \quad \text{or} \quad \frac{1}{2\sqrt{n}} - 2n$$

c)  $y = \frac{1}{v^2}$

$$y = v^{-2}$$

$$\frac{dy}{dv} = -2v^{-2-1}$$

$$\therefore \frac{dy}{dv} = -2v^{-3} \text{ or } -\frac{2}{v^3}$$

d)  $s = \frac{5}{3t^2} + 3t$

$$s = \frac{5}{3}t^{-2} + 3t^1$$

$$s' = \left(-2 \times \frac{5}{3}\right)t^{-2-1} + (1 \times 3)t^{1-1}$$

$$= -\frac{10}{3}t^{-3} + 3$$

$$\therefore s' = -\frac{10}{3}t^{-3} + 3 \quad \text{or} \quad -\frac{10}{3t^3} + 3$$

## HIGHER DERIVATIVES

We can continue to find the derivatives of a derivative. We find the

- second derivative( symbol  $y''$  or  $\frac{d^2y}{dx^2}$  ) by taking the derivative of the first derivative.
- third derivative by taking the derivative of the second derivative... etc

$$y = x^2 + 2x - 3$$

$$y' = 2x + 2$$

$$y'' = 2$$

$$y = 2x^3 + 2x - 3$$

$$\frac{dy}{dx} = 6x^2 + 2$$

$$\frac{d^2y}{dx^2} = 12x$$

### EXERCISE 61:

1. A function,  $y$ , is given as  $y = \pi^6$ , where  $\pi$  is a constant. find  $\frac{dy}{dx}$
2. Differentiate using the first principle:
  - a)  $f(x) = x^2$
  - b)  $f(x) = 5x$
  - c)  $y = 3x + 5$
3. Find the derivative of the following using the power rule:
  - a)  $f(x) = -2x^2 + x - 20$
  - b)  $u = 3v - 2v^2 + \frac{1}{v}$
  - c)  $f(x) = \frac{x^3}{3} - 4x$
  - d)  $y = \sqrt{x} - \frac{1}{3\sqrt[3]{x^5}} + 5$
  - e)  $y = 12x^5 + 3x^4 + 7x^3$
  - f)  $h(x) = x^2 - 9x + 6$
  - g)  $y = 0.4x^{10} + 3x - 1$
  - h)  $y = \frac{5}{x^3} - x + \sqrt{x}$
  - i)  $f(x) = \frac{2}{3x^9}$
4. Find  $f''(x)$  if  $f(x) = \frac{5}{x} + 2$
5. Expand and then find  $y'$  and  $y''$  for the following:
  - a)  $y = (2x - 3)(9 - x)$
  - b)  $f(x) = x^2(9 - x)$
  - c)  $y = \frac{1}{x}(3x - 2x^2)$
  - d)  $y = (\sqrt{x} + 1)(5 - x^2)$
  - e)  $y = (x^2 + 2x)(x^2 + 2x)$
  - f)  $y = x^{-1}(x + x^{-2})$

**Lesson of Life:** Education is the most powerful weapon which you can use to change the world ~ Nelson Mandela

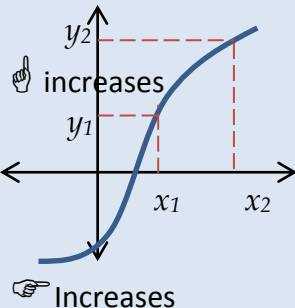
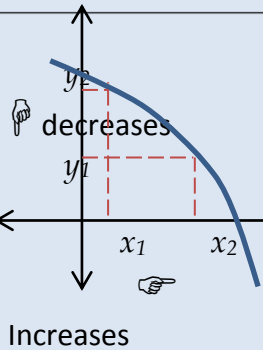
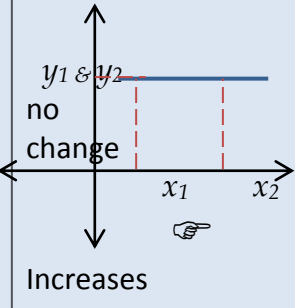
## APPLICATIONS OF DIFFERENTIATION:

### 1. SLOPE AND INTERVALS OVER WHICH FUNCTIONS ARE INCREASING OR DECREASING

**Note:** The derivative of a function gives the **gradient or slope** of a tangent to the **curve**.

To find gradient at point  $x_0$ , first differentiate the function and substitute the value of  $x_0$ .

Gradient is used to describe the increasing/decreasing function. Consider the table given:

| Curves                                                                                       | Increasing function                                                                                                                                                                                 | Decreasing function                                                                                                     | Constant function                                                                                                              |
|----------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------|
| Definition                                                                                   | A function is "increasing" when the <b>y –value</b> increases as the <b>x –value</b> increases. It is easy to see that <b><math>y = f(x)</math></b> tends to go <b>up</b> as it goes <b>along</b> . | A function is "decreasing" when the <b>y –value</b> decreases as the <b>x –value</b> increases. The graph is going down | A Constant Function is a horizontal line. The <b>y –value</b> does not change as the <b>x –value</b> increases on the x- axis. |
| General Shape                                                                                |                                                                                                                  |                                     |                                           |
| Signs of Gradients                                                                           | Positive Gradient/slope                                                                                                                                                                             | Negative Gradient/slope                                                                                                 | Zero Gradient/slope [stationary]                                                                                               |
| Theorem on Derivatives:<br>Let $f$ be a differentiable function on the interval $(x_1, x_2)$ | $f'(x) > 0$                                                                                                                                                                                         | $f'(x) < 0$                                                                                                             | $f'(x) = 0$                                                                                                                    |

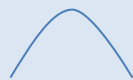
## 2. STATIONARY POINTS

**Note:** A **stationary point** of a differentiable function is a point in the domain of the function where the **derivative is zero** (equivalently, the slope of the graph at that point is zero).

At this level the focus is only on the maximum/minimum turning point.  $f'(x) = 0$

➡ The maximum turning point is when the function at point  $x_0$  is changing its direction from either increasing function to decreasing or positive gradient to negative gradient.

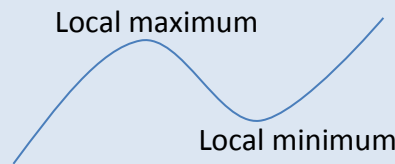
Maximum



➡ The minimum turning point is when the function at point  $x_0$  is changing its direction from either decreasing function to increasing or negative gradient to positive gradient.



Minimum



Local maximum

Local minimum

It is important to note that for a cubic function these maximum and minimum points are **local ones only**, as there are other points on the graph that are higher or lower than these turning points.

To find **Stationary Points**:

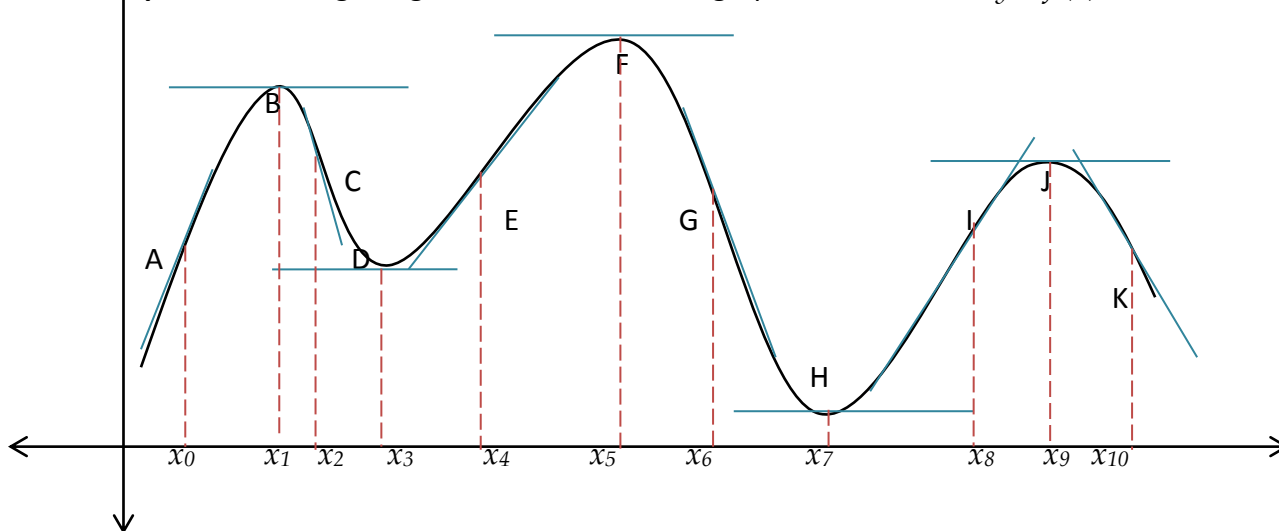
- Find  $f'(x)$ .
- Put  $f'(x) = 0$  and solve for  $x$ .
- Find the  $y$ -values by substituting the  $x$ -values you got from solving  $f'(x) = 0$  into  $y = f(x)$ .
- For cubic polynomials, the turning point with a bigger  $y$ -value is the maximum point. For parabola, look at the coefficient of  $x^2$  [ $y = \pm ax^2$ ]. If it is positive, it will have a minimum point otherwise, it will be a maximum point.

## SECOND DERIVATIVE TEST

**Note:** Suppose  $f(x)$  is a function that is twice differentiable at a stationary point  $k$ .

1. If  $f''(k) > 0$  then  $f$  has a minimum at  $k$ .
2. If  $f''(k) < 0$ , then  $f$  has a maximum at  $k$ .

**Example 1:** The diagram given below shows the graph of the function  $y = f(x)$



Identify the letters that satisfies the following conditions:

- $f'(x) > 0$
- $f'(x) < 0$
- $f'(x) = 0$

**Answers:**

- A, E, I
- C, G, K
- B, D, F, H, J

**Example 3:** A function  $f(x)$  is given by  $f(x) = x^3 - 3x$ .

- Differentiate to find  $f'(x)$ .
- Determine the  $x$  value(s) of the turning point(s) of  $f(x)$ .
- Give the coordinates of the turning point.
- State the nature of the turning points.
- Sketch the graph of  $f(x)$ .
- For what value(s) of  $x$  is  $f'(x) < 0$ ?

**Answers:**

a)  $f'(x)$

$$\begin{aligned}f(x) &= x^3 - 3x^1 \\f'(x) &= 3x^{3-1} - (1 \times 3)x^{1-1} \\ \therefore f'(x) &= 3x^2 - 3\end{aligned}$$

b)  $x$  value(s) of the turning points:

$$\begin{aligned}f'(x) &= 0 \\3x^2 - 3 &= 0 && \text{Solve for } x \\3(x^2 - 1) &= 0 \\ \frac{3}{3}(x^2 - 1) &= \frac{0}{3} \\x^2 - 1 &= 0 \\x^2 &= 1 \\\sqrt{x^2} &= \pm\sqrt{1} \\\therefore x &= \pm 1\end{aligned}$$

c). coordinates  $\Rightarrow (x, y)$

Substitute the values of  $x$  in the function  $f(x)$

$$f(x) = x^3 - 3x$$

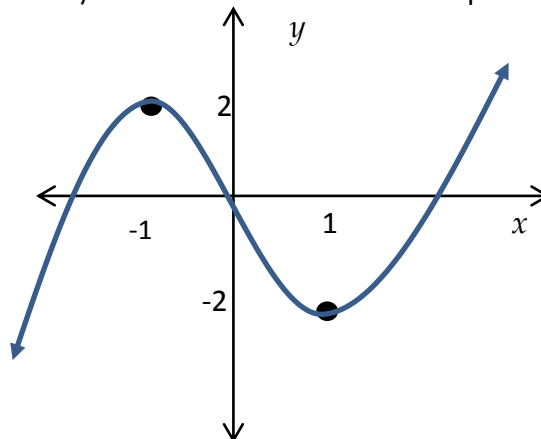
$$x = -1$$

$$\begin{aligned}f(-1) &= (-1)^3 - 3(-1) \\&= -1 + 3 \\&= 2\end{aligned}$$

$$\therefore (-1, 2)$$

$$\begin{aligned}f(1) &= (1)^3 - 3(1) \\&= 1 - 3 \\&= -2 \\ \therefore (1, -2)\end{aligned}$$

d) Nature – Method 1 From Graphs



From the graph it can be seen that  $(-1, 2)$  is a local maximum while  $(1, -2)$  is a local minimum.

**Method 2: Nature of roots using the Double Derivative Test**

$$\begin{aligned}f''(x) &= 6x \\f''(-1) &= 6 \times -1 \\&= -6 < 0 \therefore \text{Local Maximum at } (-1, 2)\end{aligned}$$

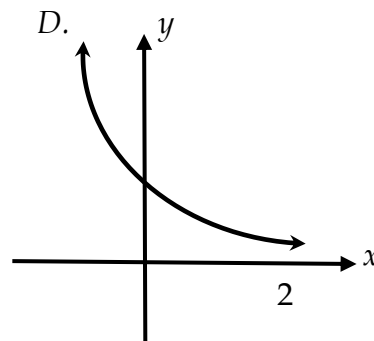
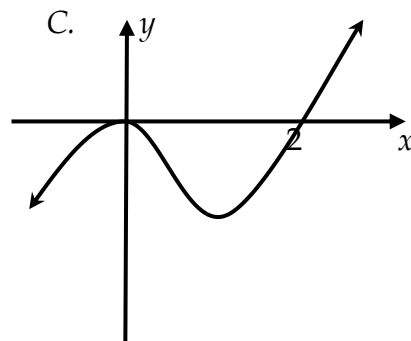
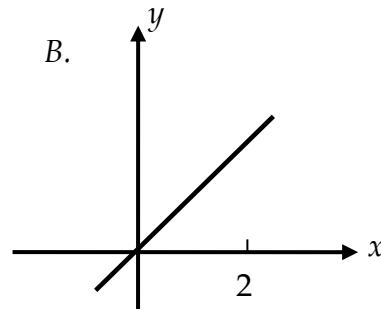
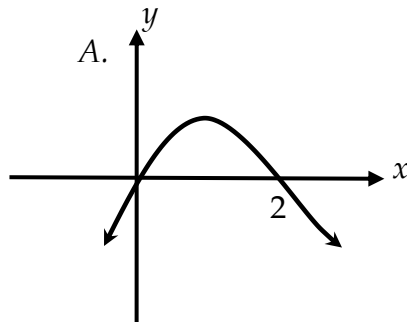
$$\begin{aligned}f''(x) &= 6x \\f''(1) &= 6 \times 1 \\&= 6 > 0 \therefore \text{Local Minimum at } (1, -2)\end{aligned}$$

e) It can be seen from the graph that  $x$  is decreasing from  $-1$  to  $1$ , thus  $-1 < x < 1$

**EXERCISE 62:**

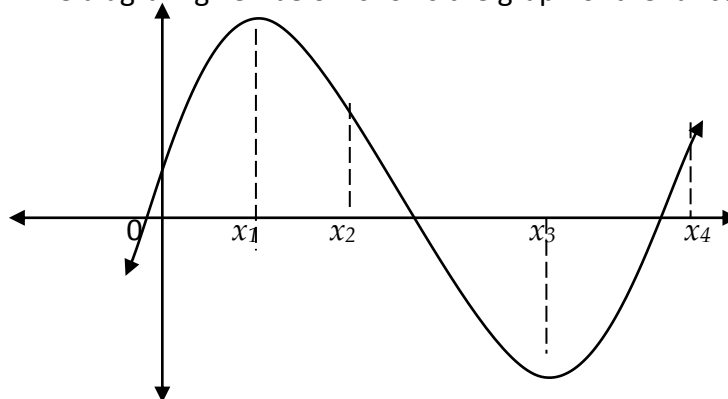
**[Multiple Choice Question]**

1. Which graph shows a function that is increasing on the interval  $0 \leq x \leq 2$ ?



**[Questions requiring working]**

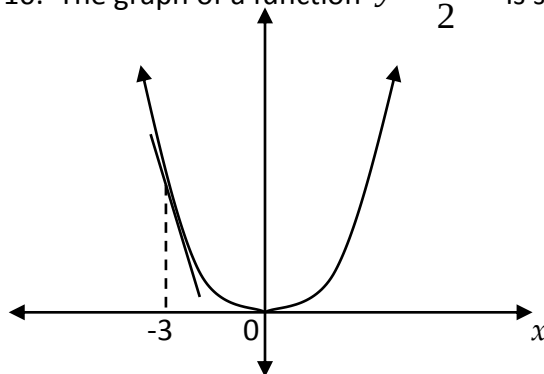
2. Determine the gradient of the curve  $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 7$  at  $x = 2$ .
3. Write down the coordinates of the point on the curve  $y = x^2 - 3x + 1$  where the gradient of the tangent to the curve is 5.
4. The diagram given below shows the graph of the function  $y = f(x)$



At which value of  $x$  is  $\frac{dy}{dx}$  the largest?

5. Given  $f(x) = x^2 - 5x - 1$ 
  - a) Find  $f'(3)$
  - b) Solve  $f'(x) = 3$
  
6. A function is given as  $g(x) = 2x - 6x^2 - 2x^3$ .
  - i) Find  $g'(x)$
  - ii) Find the coordinates of the turning points of the graph of  $g(x)$ .
  - iii) State the nature of the turning points in (ii) above.
  - iv) For what values of  $x$  is  $g(x)$  increasing?
  
7. A curve has equation  $y = 1 + 6x - 3x^2$ .
  - a) Find  $\frac{dy}{dx}$
  - b) For what values of  $x$  is  $y$  increasing?
  
8. A graph of function is given as  $f(x) = \frac{1}{3}x^3 - \frac{x^2}{2} - 12x + 6$ . Find the coordinates of the turning points of the graph of  $f(x)$ .
  
9. A function is given as  $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 7$   
 Find:
  - i)  $f'(x)$
  - ii) the coordinates of the turning points of  $f(x)$ .
  - iii) state the nature of the turning points in (ii) above.
  - iv) the value (s) of  $x$  for which  $f'(x) > 0$
  - v) the value (s) of  $x$  for which  $f'(x) < 0$

10. The graph of a function  $y = \frac{1}{2}x^2$  is shown below.



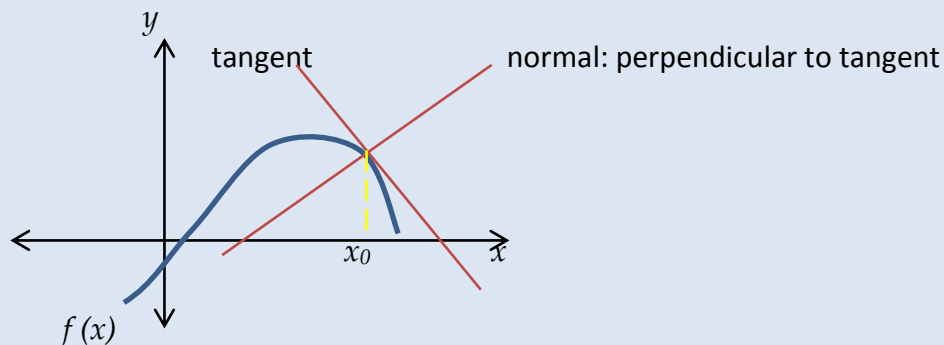
What is the gradient of the tangent to the curve at  $x = -3$ ?

**Lesson of Life:** *Mathematics is the only place where truth and beauty mean the same thing* ~ Danica McKellar

## SLOPE AND EQUATION OF TANGENT and NORMAL

**Note:** The **derivative** gives the gradient of a **curve** at any point which is also same as the gradient of a **tangent** to a curve at that point.

Consider the diagram:



To determine **the equation of a tangent** to a curve  $y = f(x)$  at the point  $x=a$  :

- Find the **derivative**  $f'(x)$  using the rules of differentiation.
- **Substitute** the  $x$ -coordinate of the given point ( $x = a$ ) into the derivative to calculate the gradient of the tangent, i.e.  $f'(a)$ .
- Substitute the **gradient** found above and the coordinates of point  **$(a, f(a))$**  into an appropriate form of the straight line equation  $y - y_1 = m(x - x_1)$ .
- Make  $y$  the subject of the formula. Express in the form  $y = mx + c$

**To determine the equation of a normal to a curve:**

The normal to a curve is the line perpendicular to the tangent to the curve at a given point.

Follow the same steps as above except for gradient, use  $m_{\text{tangent}} \times m_{\text{normal}} = -1$

**EXAMPLE :** A function is given as  $g(x) = 2x - 6x^2 - 2x^3$

i) Find  $g'(x)$

ii) Find the equation of the tangent to the curve at  $x = 2$ .

iii) Find the equation of the normal at  $x = 1$ .

**Answers:**

i)  $g'(x)$

$$g(x) = 2x^1 - 6x^2 - 2x^3$$

$$g'(x) = (1 \times 2)x^{1-1} - (2 \times 6)x^{2-1} - (3 \times 2)x^{3-1}$$

$$= 2 - 12x - 6x^2$$

$$\therefore g'(x) = 2 - 12x - 6x^2$$

ii) equation of the tangent:

- **derivative**  $g'(x) = 2 - 12x - 6x^2$
- **Substitute** the x-coordinate:  $m = g'(2) = 2 - 12(2) - 6(2)^2 = -46$
- straight line equation in the form  $y = mx + c$   
 $y - y_1 = m(x - x_1)$   
 $y - 36 = -46(x - 2)$   
 $y = -46x + 92 + 36$   
 $\therefore y = -46x + 128$   
 $x = 2;$   
 $y = g(2) = 2(2) - 6(2)^2 - 2(2)^3 = -36$   
 $\therefore (2, -36)$

iii. Equation of the normal

- **derivative**  $g'(x) = 2 - 12x - 6x^2$
- **Substitute** the x-coordinate:  $m = g'(1) = 2 - 12(1) - 6(1)^2 = -16$

$$m_{\text{tangent}} \times m_{\text{normal}} = -1$$

$$m_{\text{normal}} = \frac{-1}{-16} = \frac{1}{16}$$

- straight line equation in the form  $y = mx + c$   
 $y - y_1 = m(x - x_1)$

$$x = 1;$$
$$y = g(1) = 2(1) - 6(1)^2 - 2(1)^3 = -6$$
$$\therefore (1, -6)$$

$$\Rightarrow y - 6 = \frac{1}{16}(x - 1)$$
$$y = \frac{1}{16}x - \frac{1}{16} + 6$$
$$\therefore y = \frac{1}{16}x + \frac{95}{16}$$

### EXERCISE 63:

#### [Multiple Choice Question]

1. The slope of the tangent line to a curve  $y = f(x)$  at any point on the curve is given by  $3x^2 - 5$ . Which of the following gives the equation of the curve  $y = f(x)$  given that the curve passes through the point (3, 20)?

A.  $y = 6x$

B.  $y = 3x^2 - 14$

C.  $y = x^3 - 5x - 4$

D.  $y = x^3 - 5x + 8$

#### [Questions requiring working]

2. Find the equation of the tangent to the curve  $y = x^3$  at the point (2, 8).
3. Write the equation for both the *tangent line* and *normal line* to the curve  $y = 3x^2 - x + 1$  at the point where  $x = 1$ .
4. Find the equation of the tangent line to the curve  $y = x^2 - 1$  at the point (1,0).
5. Find the equation of the normal line to the curve  $y = 3x^2 - x + 1$  at the point (-1,5).
6. A function is given as  $f(x) = x^3 - 2x^2 + 5$  and it passes through the point (2,5).
- a) Find the gradient of  
(i) the tangent at (2,5) (ii) the normal at (2,5)
- b) Find the equation for both the *tangent line* and *normal line*.
7. For the function  $f(x) = x^2$ , Find:
- (i) the gradient of the tangent at  $x=3$   
(ii) the equation of the tangent at  $x=3$ .
8. A function is given as  $y = x^2 + x$ . Find:
- (i) the gradient of the normal at  $x = -2$   
(ii) the equation of the normal at  $x = -2$

9. What is the equation of the tangent to the curve  $f(x) = 3x^2 - 9x$  when  $x = 4$ ?

10. A function given by  $f(x) = \frac{-1}{2}x^2 + 3x + \frac{3}{2}$ .

- (i) Differentiate the function to obtain  $f'(x)$ .
- (ii) Show that the turning point occurs at  $x = 3$ .
- (iii) Determine the coordinates of the turning point.
- (iv) Determine whether the turning point is a maximum or a minimum.
- (v) Find the equation of the tangent at  $x = 2$

**Real life applications:**

**Tangent**

- A car has skidded while rounding a corner, **tangent** to the double yellow lines curve.



- Likewise, swinging a ball around in a circular motion then let go, it will fly off in a **tangent** to the circle of motion.

**Normal:**

- The spokes of a bicycle wheel are **normal** to the rim. The spokes of a wheel are placed **normal** to the circular shape of the wheel at each point where the spoke connects with the center.



Source <http://www.intmath.com/applications-differentiation/1-tangent-normal.php>

## APPLIED MAXIMUM AND MINIMUM PROBLEMS

**Note:** The terms maxima and minima refers to **extreme values** of a function, that is, the maximum and minimum values that the function attains. Applied problems in which we have to find the **maximum or minimum** are sometimes called **optimization problems**. Maximum means upper bound or largest possible quantity whereas minimum is the bound or smallest possible quantity.

Steps to find the largest/smallest possible quantity:

- List the **information**: Read the question carefully and extract the relevant information.
- Develop a function that describes the situation algebraically: These are usually the **quantities** asked for in the problem. Draw appropriate diagrams where applicable.
- Find the derivative of the function, equate to zero and solve for the quantity.
- Determine whether it is a maximum or minimum and answer the question that has been asked.

**EXAMPLE 1:** A garment factory produces T – shirts, the cost of which can be worked out by the formula:  $C(x) = x^3 - 20x^2 + 800$  and the income from the export of T – shirts is given by  $S(x) = 660x - 2x^2$ . If  $x$  is the number of T – shirts produced;

1. Write an equation for the profit expected in the sale of T – shirts made by this factory
2. How many T – shirts need to be produced to maximize the profit?

**Answers:**

1. Profit = Income – Cost or Profit = Selling price – cost price

$$\begin{aligned}P(x) &= S(x) - C(x) \\&= (660x - 2x^2) - (x^3 - 20x^2 + 800) \\&= 660x - 2x^2 - x^3 + 20x^2 - 800 \\&\therefore P(x) = -x^3 + 18x^2 + 660x - 800\end{aligned}$$

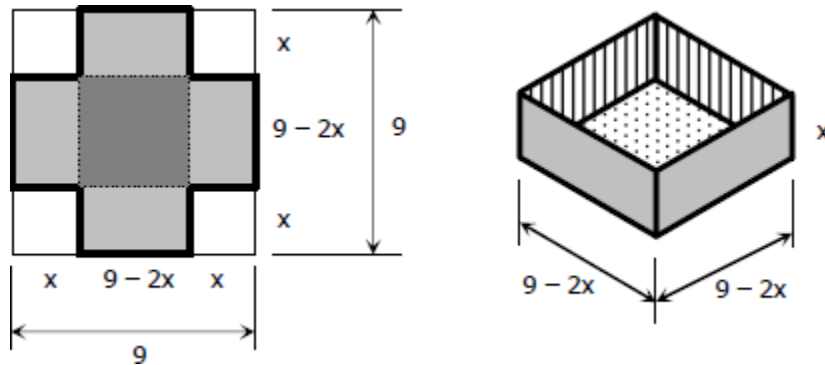
2. Maximize: Differentiate, equate to zero and solve for the quantity

$$\begin{aligned}P'(x) &= 0 \\P(x) &= -x^3 + 18x^2 + 660x - 800 \\P'(x) &= -3x^2 + 36x + 660 \\0 &= -3(x^2 - 12x + 220) \\0 &= (x - 22)(x + 10) \\x &= 22, x = -10\end{aligned}$$

Ignore the negative answer since the number of T-shirt cannot be negative.

$\therefore$  22 T-shirts

**EXAMPLE 2:** A box with an open top is to be made from a square piece of cardboard with sides 9 cm by cutting equal squares out of the corners and turning up the sides. Find the volume of the largest box that can be made in this way.



$$V = (9-2x)^2 x$$

$$V = 81x - 36x^2 + 4x^3$$

$$V' = 81 - 72x + 12x^2 = 0$$

$$V' = 3(27 - 24x + 4x^2) = 0$$

$$V' = 27 - 24x + 4x^2 = 0$$

Factorize and solve or use the quadratic formula

$$(2x - 9)(2x - 3) = 0$$

$$2x - 9 = 0 \text{ or } 2x - 3 = 0$$

$$x = 4.5 \text{ and } 1.5$$

Find volume:

$$V(4.5) = [9 - 2(4.5)]^2 (4.5)$$

$$= 0$$

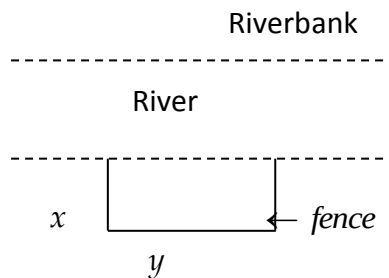
$$V(1.5) = [9 - 2(1.5)]^2 (1.5)$$

$$V = 54 \text{ cm}^3$$

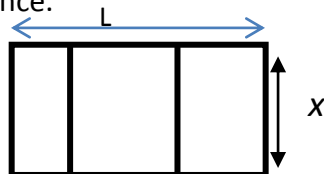
$$\therefore \text{maximum volume} = 54 \text{ cm}^3$$

#### EXERCISE 64:

1. Find the volume of the largest box with an open top that can be made by cutting equal squares out of the corners of a piece of cardboard of dimensions 15 cm by 24 cm, and then turning up the sides.
2. The sum of two numbers is 20. Calculate the maximum value of their product.
3. A rectangular farm is to be fenced using 100m of fence. One side of the farm is bounded by a river and does not need fencing.



- a) Write an expression relating  $x$ ,  $y$  and the *total length* of the fence.
  - b) Write an expression for the area enclosed in terms of  $x$  only.
  - c) Differentiate to find  $\frac{dA}{dx}$ .
  - d) Find the dimensions ( $x$  and  $y$ ) for the maximum possible area.
  - e) Find the maximum possible area.
4. A field is shown below. It is to be divided into three paddocks and fenced using 1,200 m of fence.



- a) Write an expression for the *length* ( $L$ ) in terms of  $x$ .
- b) Write an expression for the total area in terms of  $x$ .
- c) Differentiate.
- d) Find the length and width of the field that will give a **maximum** area.
- e) Find the maximum possible area.

5. The cost in dollars for a company when it produces an item of a certain product is given by

$$C(x) = 1.5 \left( 4x + \frac{160,000}{x} \right),$$

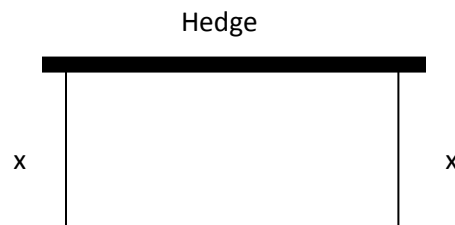
where  $x$  is the number of machines used. Find the number of machines that will make the cost a minimum.

6. Mr. and Mrs. Patel are in business, manufacturing and selling Hi Fi systems.

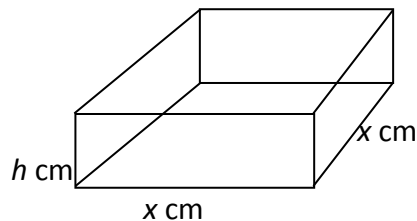
The cost of manufacture function is  $C(x) = 3000 - \frac{x^2}{3000}$ , where  $x$  is the number of products made and  $C$  is in dollars. The retail function for their income from sales is given by  $R(x) = 200x - \frac{x^2}{10}$ .

How many Hi Fi systems should they make and sell to maximize their profit?

7. A farmer is about to build a rectangular paddock alongside a dense hedge. He has only 240 m of fencing available, but he wants to enclose the largest possible area. Find the value of  $x$  for which the area is the maximum and also calculate the maximum area. (Note only 3 sides are to be fenced).

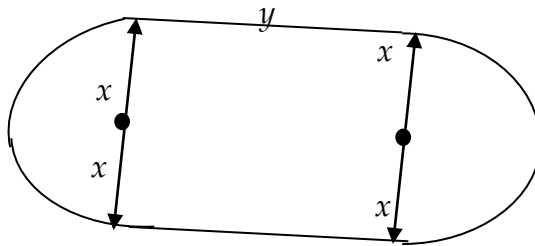


8. The rectangular box has a wire frame work with a height of  $h$  cm and a square base of side  $x$  cm. The framework is constructed of 12 pieces of wire, of total length 60 cm.



- Show that the height of the box in terms of  $x$  is  $(15 - 2x)$  cm.
- Find the volume of the box in terms of  $x$ .
- Find  $V'(x)$ .
- Calculate the maximum volume such a box can have.

9. A rectangular field is bounded on one side by a straight drain, and the other three sides by a fence whose total length is 160 m. Show that the area of the field cannot exceed  $3200 \text{ m}^2$ . (Maximum area cannot exceed  $3200 \text{ m}^2$ ).
10. A manufacturer of metallic boxes has an order for boxes that are to have a square base with no top. The volume of each box is to be 32 cubic meters.
- If the length of the boxes is  $x$  metres, show that the height is given by the expression,  $h = \frac{32}{x^2}$ .
  - Show that the surface area of the box can be expressed as  $A = x^2 + \frac{128}{x}$ .
  - Find the dimensions of the box that will minimize the amount of material needed for its construction.
11. The figure given below shows a 500 m running track around a sports field. The middle portion is a rectangle in nature while the curved paths are semicircular.



- Write an expression in terms of  $x$  &  $y$  for the perimeter of the sports field.
- Write an expression for the area,  $A(x)$  of the sports field
- Find  $\frac{dA}{dx}$
- Calculate the radius of these semicircles so that the area of the sports field is maximum.

**Lesson of Life:** *One cannot understand... the universality of laws of nature, the relationship of things, without an understanding of mathematics. There is no other way to do it ~ Richard P. Feynman*

$$\text{Life} = \int_{\text{birth}}^{\text{death}} \frac{\text{happiness}}{\text{time}} \Delta \text{time}$$



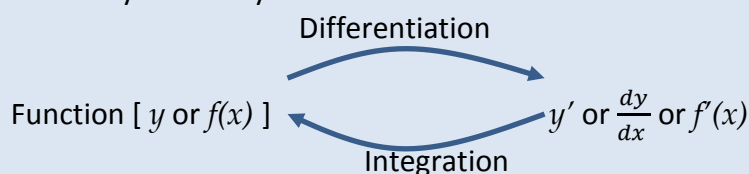
CONTENT LEARNING  
OUTCOMES

- Indefinite Integral
- Constant of integration
- Definite integral
- Applications

## INDEFINITE INTEGRAL

**Note:** Indefinite integral or antiderivative of a function ' $f$ ' is a differentiable function  $F$  whose derivative is equal to the original function  $f$ .

- It is an **opposite process** of differentiation. Also known as "anti-differentiation" or more specifically called "integration". The symbol for integration is  $\int dx$ .  $dx$  is written after the algebraic expression to show that  $x$  is the variable, and it reads as 'integration with respect to  $x$ '. if another variable is written then you integrate with respect to that variable.
- This can be stated symbolically as follows.



- The general formula:  

$$\int f(x) dx = F(x) + C \quad \text{where } C \text{ is a constant [that can take any number].}$$
- Power rule**  $f(x) = x^n$ : In differentiation, we subtract one from the power and then multiply the coefficient by the power. Since **Integration** is opposite of differentiation, so to integrate '**add one to the power and divide by the new power**'.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

- Integrating **Constant number, k**: the letter ' $k$ ' is considered here so that you don't get confused with  $C$ . After integration, just add the variable.

$$\int k dx = kx + C$$

Consider few rules below that is quite handy:

| Common Functions/<br>Rules        | Function          | Integral                | Example                                                          |
|-----------------------------------|-------------------|-------------------------|------------------------------------------------------------------|
| <b>Constant, k</b>                | $\int k dx$       | $kx + C$                | $\int 5 dx = 5x + C$                                             |
| <b>Variable</b>                   | $\int x dx$       | $x^2/2 + C$             | $\int t dt = t^2/2 + C$                                          |
| <b>Square</b>                     | $\int x^2 dx$     | $x^3/3 + C$             | $\int b^2 db = b^3/3 + C$                                        |
| <b>Multiplication by constant</b> | $\int cf(x) dx$   | $c \int f(x) dx$        | $\int 8 x^2 dx = 8 \int x^2 dx = 8/3 x^3 + C$                    |
| <b>Sum Rule</b>                   | $\int (f + g) dx$ | $\int f dx + \int g dx$ | $\int 20 + x dx = \int 20 dx + \int x dx$<br>$= 20x + x^2/2 + C$ |
| <b>Difference Rule</b>            | $\int (f - g) dx$ | $\int f dx - \int g dx$ | $\int 20 - x dx = 20x - x^2/2 + C$                               |

**EXAMPLE 1:** Calculate

1.  $\int 3000 - \frac{x^2}{3000} dx$

$$\begin{aligned} & \int 3000 - \frac{x^2}{3000} dx \\ & 3000x - \frac{1}{3000} \left( \frac{x^{2+1}}{2+1} \right) + C \\ & \Rightarrow 3000x - \frac{1 \times x^3}{3 \times 3000} + C \\ & 3000x - \frac{x^3}{9000} + C \end{aligned}$$

2.  $\int v^3 - 20v^{-2} + 800 dv$

$$\begin{aligned} & \int v^3 - 20v^{-2} + 800 dv \\ & \frac{v^{3+1}}{3+1} - \frac{20v^{-2+1}}{-2+1} + 800v \\ & \Rightarrow \frac{v^4}{4} - \frac{20v^{-1}}{-1} + 800v + C \\ & \frac{v^4}{4} + \frac{20}{v} + 800v + C \end{aligned}$$

3.  $\int \frac{5}{a^3} - a + \sqrt{a} da$

$$\begin{aligned} & \int 5a^{-3} - a^1 + a^{1/2} da \\ & \frac{5a^{-3+1}}{-3+1} - \frac{a^{1+1}}{1+1} + \frac{a^{1/2+1}}{1/2+1} + C \\ & \Rightarrow \frac{5a^{-2}}{-2} - \frac{a^2}{2} + \frac{a^{3/2}}{3/2} + C \\ & -\frac{5}{2a^2} - \frac{a^2}{2} + \frac{2}{3}a^{3/2} + C \end{aligned}$$

**EXERCISE 65:**

1. Find the following integrals:

a)  $\int x^5 dx$

b)  $\int \frac{-5}{2} x^2 dx$

c)  $\int 1 da$

d)  $\int 4x^6 dx$

e)  $\int 3x dx$

f)  $\int \sqrt{x^5} dx$

g)  $\int \frac{1}{\sqrt[3]{x^2}} dx$

h)  $\int \frac{12}{x^3} dx$

i)  $\int \left( \frac{2}{5} x^4 - \frac{3x}{2} + \frac{1}{3} \right) dx$

2. Compute [Hint: Expand first]

a)  $\int x(x+1) dx$

b)  $\int (x-4)^2 dx$

c)  $\int (x-4)(2+x) dx$

3. Integrate the following:

a)  $\int \frac{x^2 + 5x}{x} dx$

b)  $\int \frac{4x^5 - 2x}{x} dx$

c)  $\int \frac{4 - x^3}{x^2} dx$

4. Find the following integrals:

a)  $\int (x^2 - 2x + 3) dx$

b)  $\int (3x^2 + 6x - 5) dx$

c)  $\int (x^3 - \frac{1}{x^2} + \sqrt{x}) dx$

d)  $\int (-x^2 + 4x - 3) dx$

e)  $\int (x^{-3} - 50 + \frac{1}{\sqrt{x}}) dx$

## FINDING CONSTANT OF INTEGRATION

**Note:** Constant C [that can take any number] is added after the integration because the derivative of a constant is zero. It is the "**Constant of Integration**". But there could have been a constant of **any value**, hence C is a generic 'starting point' that has been added to all anti-derivatives.

This constant, C, can be found by substituting from the additional information given. Make sure you replace the value of C at the end.

**EXAMPLE 2:** A derived function is given as  $c'(x) = 60 + 20x - 3x^2$ . Write down an expression for  $c(x)$ , given that  $c(2)=0$ .

Integrate using power rule to get  $c(x)$  :

$$\begin{aligned} \int c'(x) dx &= c(x) \\ \int 60 + 20x - 3x^2 dx &= 60x + \frac{20x^{1+1}}{1+1} - \frac{3x^{2+1}}{2+1} + C \\ &= 60x + 10x^2 - x^3 + C \end{aligned}$$

Solve for C using  $c(2)=0$ , thus  $x = 2, y = 0$  :

$$c(x) = 60x + 10x^2 - x^3 + C$$

Substitute  $c(x)=y=0$

$$\begin{aligned} c(x) &= 60x + 10x^2 - x^3 + C \\ 0 &= 60(2) + 10(2)^2 - (2)^3 + C \\ 0 &= 152 + C \\ C &= -152 \end{aligned}$$

Hence the equation is

$$c(x) = 60x + 10x^2 - x^3 - 152$$

**Example 3:**  $f'(x) = 2x - 9$  and  $f(1) = 5$ . Find  $f(x)$ .

$$\begin{aligned}f(x) \text{ can be found by integrating } f'(x). \text{ i.e.: } f(x) &= \int f'(x) \\f(x) &= \int 2x - 9 dx \\&= x^2 - 9x + c\end{aligned}$$

Now if  $f(1) = 5$  means when  $x=1$ ,  $f(x) = 5$

$$\Rightarrow 5 = 1^2 - 9 \times 1 + c$$

$$\Rightarrow 5 = 1 - 9 + c$$

$$c = 13$$

$$\text{Therefore, } f(x) = x^2 - 9x + 13$$

#### EXERCISE 66:

1. Find an expression for  $y$  given  $\frac{dy}{dx} = 7x^2$  and when  $x = 3$ ,  $y = 10$ .
2. Find  $f(x)$  if  $f'(x) = 2x - 3$  and  $f(4) = 7$ .
3. Given that  $v'(t) = 2t + 3t^2$ , find the expression for  $v(t)$  if  $v(0) = 30$ ?

### DEFINITE INTEGRAL

**Note:** So far we have been finding indefinite integrals.

The *definite integral* of  $f(x)$  between  $a$  (lower limit) and  $b$  (upper limit) can be defined as follows:

$$\int_a^b f(x) dx = \left[ F(x) \right]_a^b = F(b) - F(a)$$

Steps:

- Integrate the function  $f(x)$ . You may ignore the constant  $C$  since it will get cancelled.
- Substituting two values ( $a$  and  $b$ ) into the anti-derivative. Start with upper limit ( $b$ )
- Find the difference

**Example 1:** Evaluate  $\int_{-1}^2 x(2x+1) dx$

Integrate the function  $f(x)$ :

$$\int_{-1}^2 x(2x+1) dx = \int 2x^2 + x dx = \frac{2x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} = \frac{2x^3}{3} + \frac{x^2}{2} + C$$

Substitute the two values ( $a$  and  $b$ ):

$$\begin{array}{ll} b=2 & a=-1 \\ \frac{2b^3}{3} + \frac{b^2}{2} & \frac{2a^3}{3} + \frac{a^2}{2} \\ \frac{2(2)^3}{3} + \frac{(2)^2}{2} & \frac{2(-1)^3}{3} + \frac{(-1)^2}{2} \\ \frac{16}{3} + \frac{4}{2} & -\frac{2}{3} + \frac{1}{2} \\ 7\frac{1}{3} & -\frac{3}{6} + \frac{3}{6} = -\frac{1}{6} \end{array}$$

Find the difference:

$$7\frac{1}{3} - -\frac{1}{6} = \frac{22}{3} + \frac{1}{6} = 7.5$$

**In just one step:**

$$\int_{-1}^2 x(2x+1) dx = \left[ \frac{2x^3}{3} + \frac{x^2}{2} \right]_{-1}^2 = \left[ \left( \frac{2(2)^3}{3} + \frac{(2)^2}{2} \right) - \left( \frac{2(-1)^3}{3} + \frac{(-1)^2}{2} \right) \right] = 7.5$$

**Example 2:** The rate of rainfall per day, measured in cm,  $x$  days after the beginning of the year, is given as  $R'(x) = 0.00002(6511 + 366x - x^2)$ . Find  $R(x)$  for the first 180 days of the year?

$$\begin{aligned} R'(x) &= 0.00002(6511 + 366x - x^2) \\ \int_0^{180} R'(x) dx &= \int_0^{180} 0.00002(6511 + 366x - x^2) dx = 0.00002 \int_0^{180} (6511 + 366x - x^2) dx \\ R(x) &= 0.00002 \int_0^{180} (6511 + 366x - x^2) dx \\ &= 0.00002 \left[ 6511x + \frac{366}{2}x^2 - \frac{x^3}{3} \right]_0^{180} \\ &= 0.00002 \left[ \left( 6511(180) + \frac{366}{2}(180)^2 - \frac{(180)^3}{3} \right) - \left( 6511(0) + \frac{366}{2}(0)^2 - \frac{(0)^3}{3} \right) \right] \\ &= 103.14 - 0 \\ &= 103.14 \text{ cm} \end{aligned}$$

Thus 103.14 cm.

**EXERCISE 67:**

1. Calculate the following definite integrals:

a)  $\int_2^4 s \, ds$

b)  $\int_{-3}^0 (2x + 3x^2) dx$

c)  $\int_{-3}^2 p^2 + 2p - 3 \, dp$

d)  $\int_1^2 \frac{3}{x^2} \, dx$

e)  $\int_1^4 (x^{-3} - 50 + \frac{1}{\sqrt{x}}) dx$

f)  $\int_2^3 (x-2)^2 dx$

g)  $\int_1^5 \frac{4-x^3}{x^2} dx$

h)  $\int_{-2}^1 (x+1)(x-1) dx$

i)  $\int_1^2 \frac{1}{x^2} dx$

j)  $\int_{-1}^2 \left( \frac{3x^3 + 10x^2}{x} \right) dx$

k)  $\int_1^{16} \frac{4}{3\sqrt{x}} \, dx$

l)  $\int_0^4 \frac{6}{x^2} \, dx$

2. Solve for k when  $\int_0^k 3\sqrt{x} dx = 54$

3. The expression for the marginal cost (\$) is given as  $c'(x) = 6000 + 200x - 3x^2$ , where x is the number of units. What is the increase in cost  $c(x)$  when the number of items is increased from 20 to 30 units?

4. A company determines that its production facility follows the learning curve  $f'(x) = 1000x^{-2}$ , where  $f'(x)$  is the rate of labour hours required to assemble the  $x^{th}$  unit. What is the number of labour hours required  $f(x)$  to produce between 25 and 50 units.

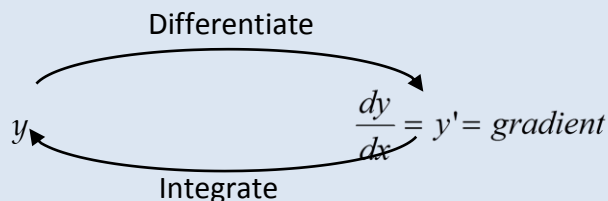
**Lesson of Life:** *The object of education is to prepare the young to educate themselves throughout their lives* ~ Robert M. Hutchins

## APPLICATIONS OF INTEGRATION :

### 1. EQUATION OF CURVES

**Note:** One use of integration is to calculate the equation of a curve when a gradient function is known. If the coordinates of a point on a curve is given, then the value of constant C can be worked out.

Since we are working with equations of curves in the form  $y = \dots$ , it is often more convenient to express the gradient function in the form  $\frac{dy}{dx} = \dots$



**EXAMPLE 1:** A curve has a gradient function given by:  $\frac{dy}{dx} = x^2 - 2x + 1$ . Find the equation of the curve if it passes through the point (1,6).

Integrate using power rule to get  $y$  :

$$y = \int x^2 - 2x + 1 \, dx$$
$$\Rightarrow \frac{x^{2+1}}{2+1} - 2 \frac{x^{1+1}}{1+1} + 1x + C$$
$$y = \frac{x^3}{3} - 2 \frac{x^2}{2} + x + C$$

Solve for C using (1,6), thus substitute  $x = 1, y = 6$  :

$$y = \frac{x^3}{3} - 2 \frac{x^2}{2} + x + C$$
$$6 = \frac{1^3}{3} - 1^2 + 1 + C = \frac{1}{3} + C$$
$$\therefore C = 5 \frac{2}{3}$$

Hence the equation is

$$y = \frac{x^3}{3} - x^2 + x + 5 \frac{2}{3}$$

**Example 2:** A curve goes through the point (-2,3) and has a derived function given by  $\frac{dy}{dx} = 3x^2 - 8x + 1$ . What is the equation of this curve?

$$\frac{dy}{dx} = 3x^2 - 8x + 1$$

Integrate:

$$y = \int (3x^2 - 8x + 1) dx$$

$$y = x^3 - 4x^2 + x + c$$

Now find  $c$  (when  $x = -2, y = 3$ )  $\Rightarrow 3 = (-2)^3 - 4(-2)^2 + -2 + c$  (solve for  $c$ )

$$c = 29$$

$$\text{Therefore, } y = x^3 - 4x^2 + x + 29$$

### EXERCISE 68:

1.  $\frac{dy}{dx} = 3x^2$  and  $(y = 4, x = 1)$ . Find an expression for  $y$  in terms of  $x$ .
2. The gradient of a tangent to a curve which passes through the point  $(2,4)$  is given by  $2x$ . Find the equation of the curve.
3. The expressions for the gradient of a tangent line to the graph of a certain function is given by  $\frac{dy}{dx} = 2x + 3$ . If the graph of the function  $y$  passes through the point  $(1,8)$ , calculate the function  $y$ .
4. A curve has a gradient function given by:  $\frac{dy}{dx} = 3x^2 - 1$ . Find the equation of the curve if it passes through the point  $(1,2)$ .
5.  $\frac{dy}{dx} = 4x^3 - 2x + 1$  and  $y = 3$  when  $x = 0$ . Find  $y$  as a function of  $x$ .
6. Economists have borrowed the idea of the derivative from calculus.  $C(x)$  is the cost of producing  $x$  units, while  $C'(x)$  is the marginal cost. If marginal cost is given as  $C'(x) = \frac{3}{2} \left( 4x + \frac{16000}{x^2} \right)$ , calculate the cost of production if 50 units are being produced.  $[C(0)=0]$

**Lesson of Life:** *Mathematicians are like Frenchmen: whenever you say something to them, they translate it into their own language, and at once it is something entirely different* ~ Johann Goethe

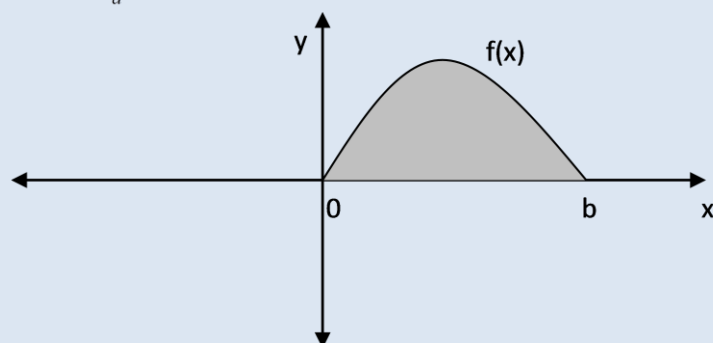
## 2. DEFINITE INTEGRALS AND AREA

**Note:** The process of integration is equivalent to finding the area between a curve and the x-axis.

### 1. Above the x - axis

Graphically the shaded Area under the curve bounded above by the x-axis from  $a$  to  $b$  will have general formula:

$$\text{Area} = \int_a^b f(x)dx = \left[ F(x) \right]_a^b = F(b) - F(a)$$

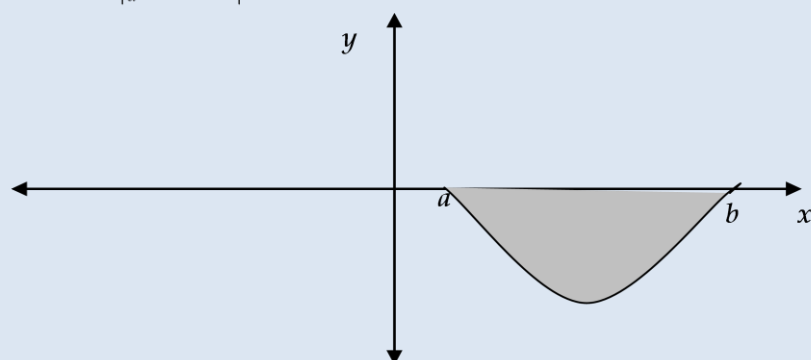


### 2. Below the x - axis

If the graph is below the x- axis , the definite integral is negative of the shaded area.

General formula:

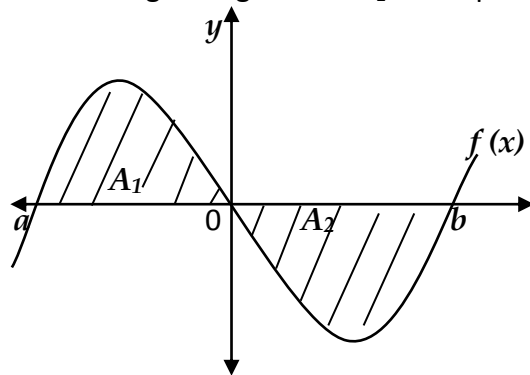
$$\text{Area} = \left| \int_a^b f(x)dx \right| = \left| \left[ F(x) \right]_a^b \right| = \left| F(b) - F(a) \right|$$



### 3. Using areas to find definite integrals

To find the **definite integrals** using areas, the value will be **negative** if the area is **below the x – axis** and **positive** if the area is **above the x-axis**.

**Example 1:** The diagram below shows the graph of a function  $y = f(x)$ . If the area of labeled region is given as:  $A_1 = 10$  square units and  $A_2 = 15$  square units,



a) find the total shaded area.

$$10 + 15 = 25$$

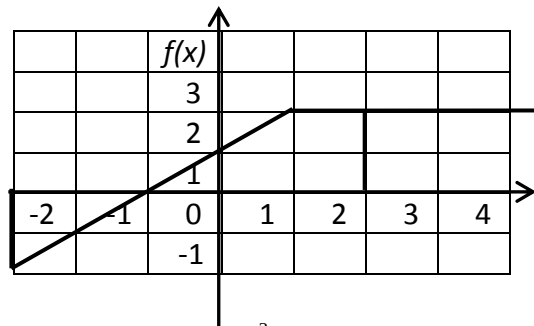
b) find

$$\int_a^b f(x) dx.$$

Since the question is on definite integral, you need to take negative of the area if it is below the x-axis

$$\int_a^b f(x) dx = A_1 + -A_2 = 10 - 15 = -5$$

**Example 2:** The graph of the function  $f(x)$  is given below.



What is the value of  $\int_{-2}^2 f(x) dx$ ?

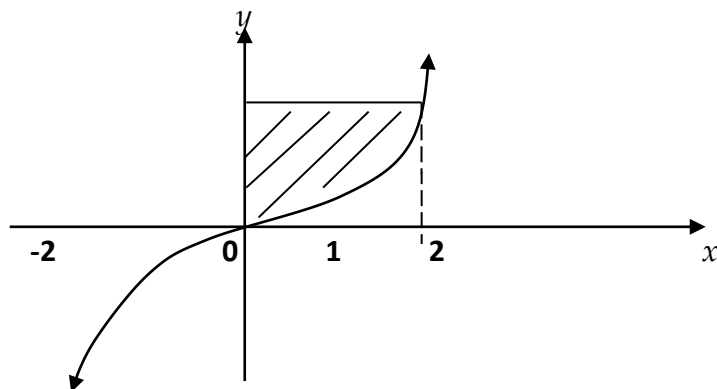
[Count the squares]

Definite integral below axis:  $\int_{-2}^{-1} f(x) dx = -2 \text{ unit}^2$

Definite integral above the axis:  $\int_{-1}^2 f(x) dx = 4 \text{ unit}^2$

$$\therefore \int_{-2}^2 f(x) dx = -2 + 4 = 2 \text{ unit}^2$$

**Example 3:** The diagram below shows the graph of  $f(x) = x^3$



- Evaluate  $\int_0^2 f(x)dx$
- Given that  $\int_{-2}^2 f(x)dx = 0$ , what is the value of  $\int_{-2}^0 f(x)dx$ ?
- Use your answer in part (a) or otherwise, to find the area of the shaded region.

**Answer:**

$$a) \int_0^2 f(x)dx = \int_0^2 x^3 \cdot dx = \left[ \frac{x^{3+1}}{3+1} \right]_0^2 = \left[ \frac{x^4}{4} \right]_0^2 = \left[ \left( \frac{2^4}{4} \right) - \left( \frac{0^4}{4} \right) \right] = 4 - 0 = 4$$

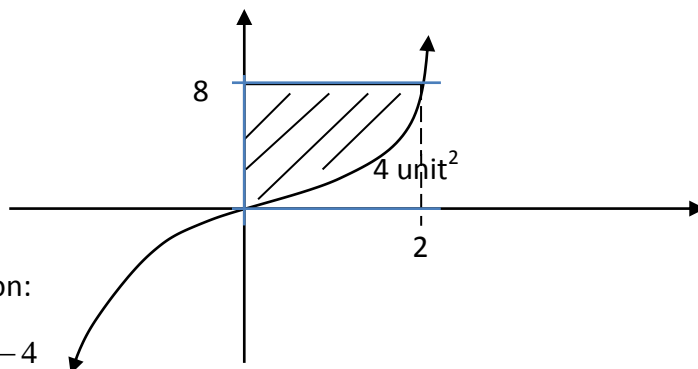
- Consider integrals above (positive) and below (negative)

$$\int_{-2}^2 f(x)dx = \int_{-2}^0 f(x)dx + \int_0^2 f(x)dx$$

$$0 = \int_{-2}^0 f(x)dx + 4$$

$$\therefore \int_{-2}^0 f(x)dx = -4$$

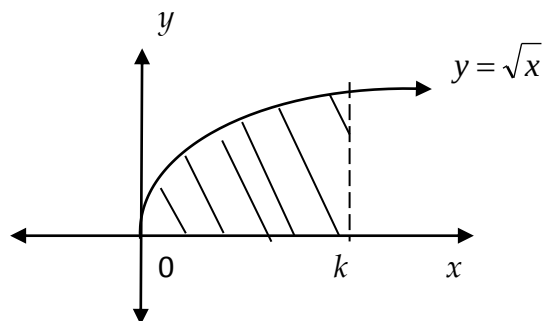
- 



To find area of the shaded region:

$$\begin{aligned} \text{Area} &= A_{\text{rectangle}} - \int_0^2 f(x)dx = l \times w - 4 \\ &= 2 \times 8 - 4 = 12 \text{ unit}^2 \end{aligned}$$

**Example 4:** The graph of the function  $y = \sqrt{x}$  is shown below.



Find  $k$  so that the area of the shaded region is 18 square units.

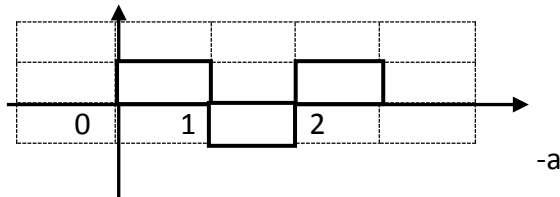
$$\begin{aligned}
 \text{Area} = 18 &= \int_0^k \sqrt{x} dx = \int_0^k x^{1/2} dx = \left[ \frac{x^{1/2+1}}{1/2+1} \right]_0^k = \left[ \frac{x^{3/2}}{3/2} \right]_0^k = \left[ \frac{2}{3} x^{3/2} \right]_0^k \\
 18 &= \left( \frac{2}{3} k^{3/2} \right) - \left( \frac{2}{3} (0)^{3/2} \right) \\
 18 &= \frac{2}{3} k^{3/2} - 0 \\
 18 &= \frac{2}{3} k^{3/2} \\
 18 \times 3 &= 2k^{3/2} \\
 \frac{54}{2} &= k^{3/2}
 \end{aligned}$$

Solve for  $k$

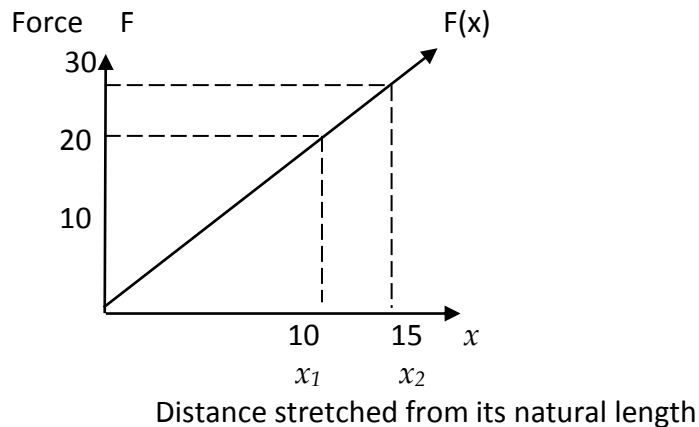
$$\begin{aligned}
 \frac{54}{2} &= k^{3/2} \\
 27 &= k^{3/2} \\
 27^1 &= k^{3/2} \\
 27^{1 \times 2/3} &= k^{3/2 \times 2/3} \\
 27^{2/3} &= k^1 \\
 k &= 9
 \end{aligned}$$

### EXERCISE 68:

1. The graph below represents the function  $g(x)$ . Each block has area of 1 unit<sup>2</sup>, that is  $\int_0^1 g(x) = 1 \text{ unit}^2$ , evaluate  $\int_0^3 g(x)$  ?

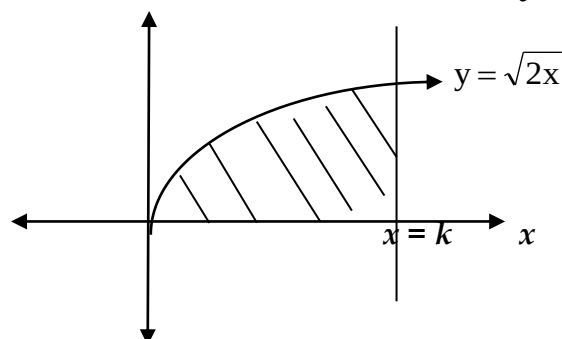


2. The work done  $W$ , in stretching a spring  $x_1$  units from its natural length to  $x_2$  units from its natural length is given as  $W = \int_{x_1}^{x_2} F \cdot dx$  where  $F$  is the force required to stretch the spring. The graph below shows how the force varies with the distance the spring stretched.



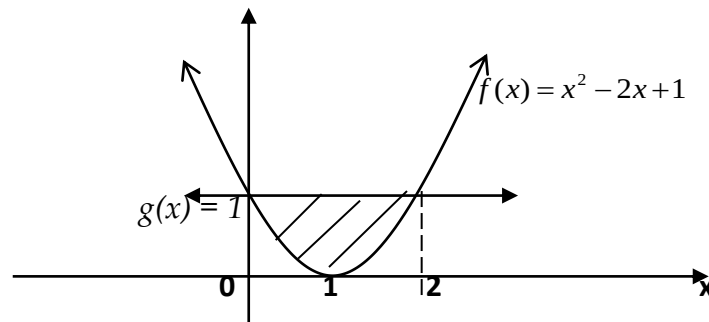
Calculate the work done to stretch the spring 10 cm from its natural length to 15 cm from its natural length?

3. The diagram given below shows the region bounded by the graph of  $y = \sqrt{2x}$ , the  $x$ -axis & the vertical line  $x = k$

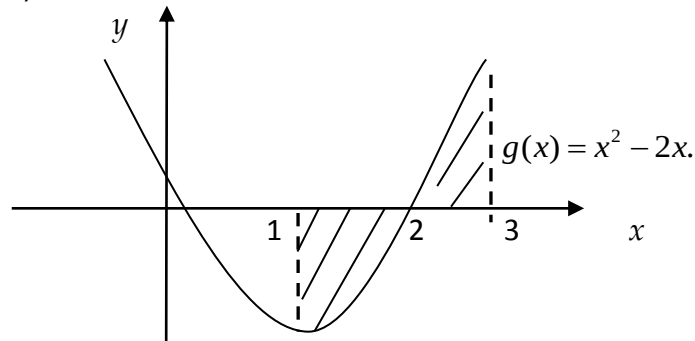


Given that the area of the region is 12 square units, determine the value of  $k$ .

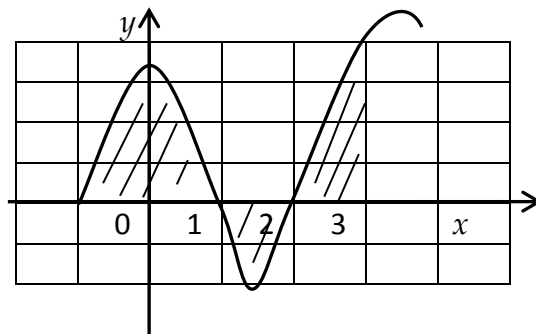
4. The diagram below shows the graph of  $f(x) = x^2 - 2x + 1$  and the line  $g(x) = 1$  intersects at point  $(0,1)$  and  $(2,1)$ . (Diagram not to scale)



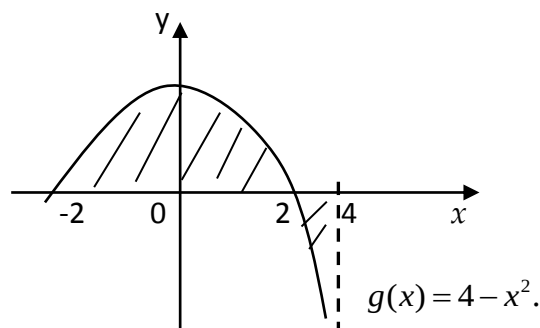
- write an expression that could be used to find the area of the shaded region.
  - Hence find the area of the shaded region
5. The diagram below shows the graph of the function  $g(x) = x^2 - 2x$ . (Diagram not to scale)



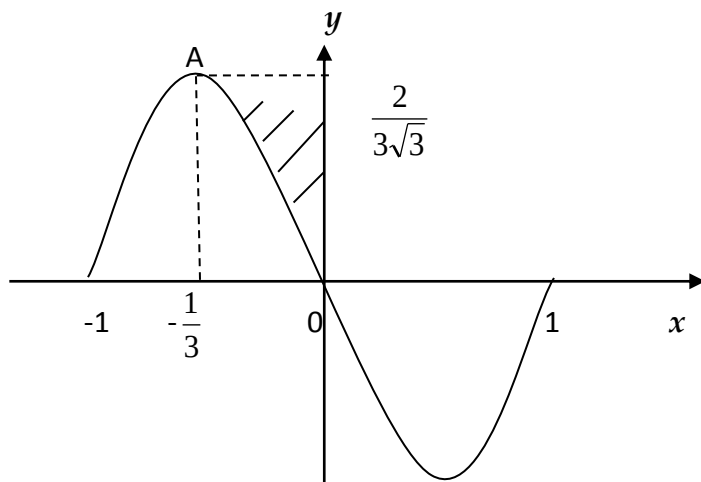
- Evaluate  $\int_1^2 g(x) dx$
  - Write an expression in terms of  $g(x)$  that will help find the area of the shaded region.
  - Hence, find the area of the shaded region
6. The graph below represents the function  $f(x)$ . Write the expression for the area of the shaded region.



7. The diagram below shows the graph of the function  $g(x) = 4 - x^2$ . (Diagram not to scale)



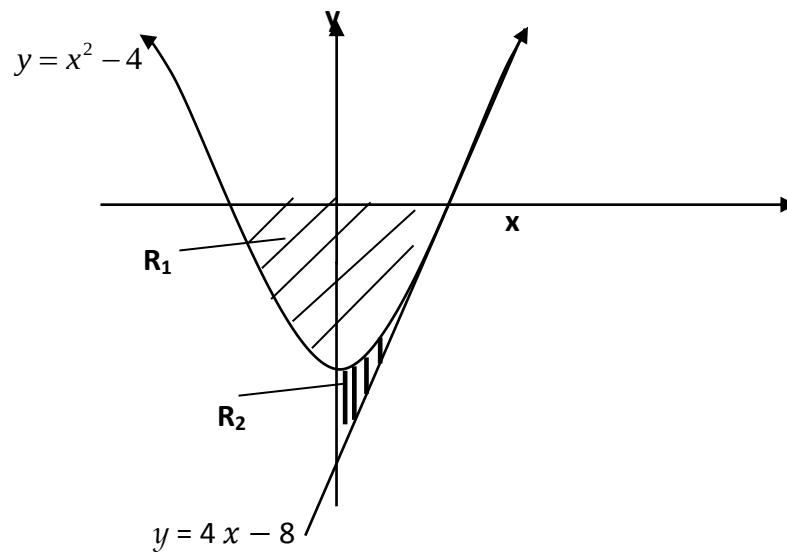
- Evaluate  $\int_2^4 g(x)dx$ .
  - Given that  $\int_{-2}^2 g(x)dx = 10\frac{2}{3}$ , find  $\int_{-2}^0 g(x)dx$ .
  - Hence, find the area of the shaded region.
8. The diagram below shows the graph of  $f(x) = x(x+1)(x-1)$



The coordinates of point A are  $A\left(-\frac{1}{3}, \frac{2}{3\sqrt{3}}\right)$

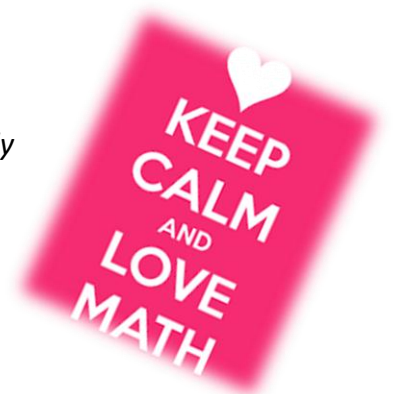
- Evaluate  $\int_0^1 f(x)dx$
- What is the value of  $\int_{-1}^0 f(x)dx$
- Given that  $\int_{-\frac{1}{3}}^0 f(x)dx = \frac{8}{81}$ , find the area of the shaded region.

9. The diagram below shows the graph of the quadratic equation  $y = x^2 - 4$  & straight line  $y = 4x - 8$ .



- Evaluate  $\int_0^2 (x^2 - 4) dx$ .
- Hence, determine the area of the region  $R_1$
- Determine the area of the region  $R_2$
- Determine a value of  $k > 0$  such that  $\int_0^k (x^2 - 4) dx = 0$ .

**Lesson of Life:** *Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture ~ Bertrand Russell*



## More Mathematical quotes for encouragement

"The whole purpose of education is to turn mirrors into windows."

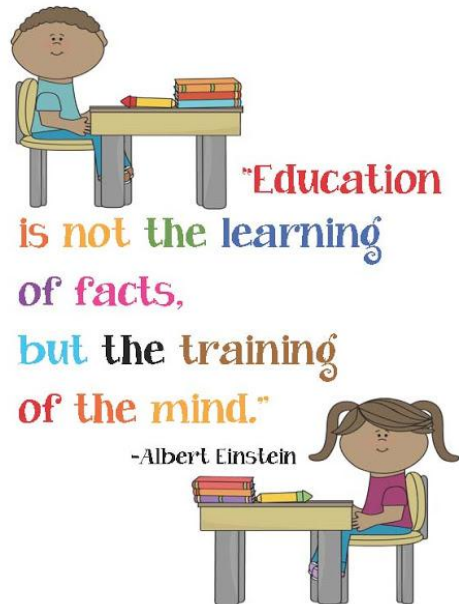
Sydney J. Harris

"Spoon feeding, in the long run teaches us nothing but the shape of the spoon."

E. M. Forster

"A mathematician is a blind man in a dark room looking for a black cat which isn't there."

Charles R. Darwin



Clip art by Laura Strickland

## HOW TO STUDY FOR A MATH EXAM

**When studying for a math exam, you have two main goals:**

- Learn the material so you can do well on the exam.
- Learn the material well enough so you will still know it next time!!!

**TIP #1:** Have all memorizing done *a couple of days before* the exam.

**TIP #2:** Use **flash cards** for memorization of formulas and rules!!!

### 1) Starting out:

Look over the Math notes. REWORK EXAMPLES!! Look over homework.

### 2) Make an exam for yourself (or better yet, for a study partner):

This should be done at least TWO (2) days before the exam - NOT the night before or you'll freak yourself out!

**NOTE:** It is extremely important that you be able to do the problems without knowing what section they came out of!! Be sure to mix the problems up when you are practicing!

### 3) Restudy:

Go back over what you had trouble with on your practice exam. This is the stuff that you didn't absorb well enough from just doing your homework.

### 4) The afternoon before exam day:

Read through your notes and think! Work some problems and review memorizing.

### 5) The night before exam day:

Do something fun and relaxing ----- But not *too* much fun! ☺

### 6) One hour before exam:

Glance over flash cards and don't talk to classmates -- They may say something to confuse you or make you nervous. 🙅

<http://www.coolmath.com/studytip>

## THROUGHOUT THE ACADEMIC YEAR

- Classes

**Attend every class.** Listen and pay attention to the material.

**Ask your teacher any questions that you might have before the day of the exam.** Highlight any questions you are unsure of and take them to your teacher who will always be happy to help you with the problem.

- Studying

**Read the textbook:** Make sure you read all of the text and not just the examples.

**Do Homework problems:** Most classes have assigned problems that the teacher feels are most useful. A lot of exam problems are **extremely similar** to homework problems.

**Keep your assessment papers:** File the check papers and worksheets in a plastic wallet or binder. Use them when revising.

**Try to find out various ways to tackle a certain problem:** For example, with systems of equations, you can solve them by substitution, elimination, or graphing.

- Revising

**Start studying 2 months before the exam:** Do not wait till the last minute. As for the day before the exam, do not be stressed and just relax. Clear your mind when you sleep and you will definitely do well.

**Join a study group:** Different people see concepts in different ways. Something that you have difficulty understanding may come easily to a study partner.

**Have someone make up problems for you to work out:** Don't try to create your own study sheet since you're not challenging yourself enough.

**Sleep for 7-9 hours to keep your mind fresh and perform calculations mentally.**

- During the test

**Relax.** Start by doing the easiest problems first. That way, you can have more time focusing on the harder problems.

Do not look at the time too often - that reduces your time doing the actual test.

**For multiple choice questions:** read each question carefully and choose the **best answer** from the options given. If a question is taking a very long time, go on to the next question and **go back** to the one you missed at the **end of the time**.

### More Tips

- Solve problems. In this way, you have the tendency to understand and realize the **formulas** and the **given problems**.
- Make sure to drink **lots of water** and have a small snack before you study. This will stimulate your brain and will help you memorize and work on your math concepts.
- Start studying while you still have time to go to a teacher for answers if you need to.
- If math is something you **find boring** and not worth studying for, give yourself **incentives** to finish problems.
- Keep calm and think positive, be confident that you can do it.
- Do **not rely on your teacher** to make you understand a concept or a problem. **Instead, do it all by yourself**, start to finish. Some questions are so tricky, they almost always have to be memorized, so mark them and revise them again and again before an exam so that it is well set in your mind.
- Try to **enjoy math**. Feel happy and satisfied when you manage to finish a problem and then proceed to the next sum.

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